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Disruptions in One-Warehouse Multiple-Retailer Systems

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We study two-echelon distribution systems (also known as one-warehouse, multiple-retailer (OWMR) systems) subject to supply disruptions. We propose algorithms to find the optimal or near-optimal stocking levels of all the locations in the system by assuming periodic review base-stock policies, deterministic demands at the retailers and non-overlapping disruptions at the supply processes of the warehouse and the retailer. This is the first paper to consider OWMR systems with all locations keeping inventory and all locations subject to supply disruptions. We show how supply disruptions at different parts of the network affect inventory decisions and we quantify the effects of ignoring the disruptions at different parts of the supply chain. Our results suggest that companies should work on reducing the duration of supply disruptions instead of trying to prevent them. In addition, if they choose to do nothing to prevent the consequences of some of the disruptions, these should not be the ones happening close to the customers.

Key words: supply disruptions; one-warehouse, multiple-retailer systems; heuristics

1. Introduction

Supply chains are subject to many types of uncertainties. Most of the early research in inventory theory concentrates purely on demand uncertainty, but there has been a growing tendency to develop models that also take supply uncertainty into consideration. Supply uncertainty happens when a company's suppliers or its own facilities cannot deliver the required quantity at the required time.

Typically, the literature divides supply uncertainty into three main categories. The first one is *disruptions*. When the supply process of a company is disrupted, it cannot receive any items that are supposed to be delivered by this supply process until the disruption is over. The second form of supply uncertainty is *yield uncertainty*, which occurs when the amount provided by the supplier is a random variable which either depends on the ordered quantity or is independent of it. The third type of supply uncertainty is *stochastic leadtimes*. When its leadtime is stochastic, the company receives the exact amount it ordered but waits a random amount of time until it is delivered. In this study, we consider the first type of supply uncertainty, i.e., disruptions.

Disruptions can happen due to a wide range of reasons like natural disasters, labor strikes, terrorist attacks, and so on. Companies whose supply processes are affected by disruptions may experience delays in transportation and disfunction in some of their facilities, which may result in shortages in inventories. Although firms can take measures to prevent them, some disruptions are inevitable. Hence, in order to avoid the drastic impact of these disruptions, firms need to protect against them. Unfortunately, the methods applied to prevent or mitigate the effects of demand uncertainty may not work when there is uncertainty in the supply process (Snyder and Shen 2006).

There are numerous real-world examples and analytical studies showing the effect of planning against or ignoring supply disruptions. These effects may last for years after disruptions are over. Hendricks and Singhal (2003, 2005a,b) report many minor and major disruptions and point out that even very minor ones can have substantial effects. The authors also report that companies did not quickly recover from these negative consequences. Their findings suggest that, even if they can't be prevented, it is crucial for companies to plan ahead of time to minimize the damage caused by supply disruptions. There are multiple tactics that companies can choose from for managing the risk of disruptions (Tomlin 2006). The most appropriate tactic for mitigating the effect of disruptions depends both on the nature of disruptions and the long-term goals of companies. In this study, we assume that the company under consideration uses inventory mitigation as its tactic.

The effect of disruptions in multi-echelon inventory systems can be more severe than in single-location systems. In multi-echelon systems, a disruption in the supply process of a location can impact all the other locations. Some of them may not receive inventories they request and the others may need to carry excess inventories that they originally planned to send if disruptions did not happen. For effective management of multi-echelon inventory systems subject to disruptions, all locations need to develop strategies for protecting against the damages caused by disruptions. In general, the analysis of multi-echelon supply chains is harder than the analysis of single-stage systems. Considering the possibility of disruptions makes the problem even harder. In the literature, there are very few studies that consider disruptions in multi-echelon systems. In fact, this is the first study to consider one-warehouse multiple-retailer (OWMR) systems with all the locations keeping inventory and all the locations subject to supply disruptions.

Assuming constant disruption and recovery probabilities, deterministic customer demands and linear unit holding and backordering costs, we determine expressions for the average cost functions when retailers are identical and non-identical. The optimum stocking policy for this system is unknown, so we assume that each location follows a base-stock policy for replenishing its inventories. We develop expressions (exact or approximate) for the optimal base-stock levels of all the locations. Via numerical analysis, we analyze the effects of ignoring disruptions at the different parts of the system and we study the sensitivity of the optimal inventory decisions to the disruption and recovery probabilities.

The structure of this paper is as follows. In Section 2, we review relevant literature in the field of supply disruptions. In Section 3, we analyze the effects of supply disruptions on optimal inventory decisions when all the retailers are identical. In Section 4, we relax the assumption of identical retailers. We present the results of our computational analysis, including sensitivity analysis to demonstrate the effect of different system parameters on the optimal order-up to levels and optimal average costs, in Section 5. In addition, we provide some managerial insights.

2. Literature Review

In this section, we briefly discuss the inventory models proposed in the literature for multi-echelon systems subject to supply disruptions. We refer the reader to Vakharia and Yenipazarli (2008) and Snyder et al. (2012) for comprehensive reviews of the literature on supply disruptions. In addition, Atan and Snyder (2012a) and Atan and Snyder (2012b) review inventory models with disruptions, Yano and Lee (1995) review yield uncertainty and Nahmias (1979) reviews stochastic leadtimes.

For disruption-safe serial systems with constant replenishment leadtimes and linear holding and backordering costs, Clark and Scarf (1960) prove the optimality of echelon base-stock policies and provide a recursive algorithm to solve for the base-stock levels of all the locations in the system. Introducing supply disruptions at all nodes of this system, DeCroix (2012) proves the optimality of echelon base-stock policies for Bernoulli disruptions. For the same system with disruptions that are governed by a DTMC, the author proves the optimality of state-dependent base-stock policies and shows that the base-stock level of a node depends on the state of the disruption process at the node itself and all the remaining downstream locations.

Schmitt et al. (2010a) consider a distribution system but allow inventory to be held only at one echelon. This effectively reduces the system to copies of single-echelon systems, unlike our model. The authors consider two scenarios. The first one is a centralized system in which only the warehouse stocks inventories and the second one is a decentralized system in which only the retailers stock inventories. Assuming that supply disruptions affect only the locations keeping inventory, the authors prove that the variance of the cost is higher for the centralized system while the expected costs are the same for the two systems. This is called *the risk-diversification effect.* Hence, in contract to distribution systems subject to demand uncertainty, decentralized systems are preferable for systems subject to supply disruptions.

Our paper contributes to the literature by being the first to consider OWMR systems with all locations keeping inventory and all locations subject to supply disruptions. By making a few simplifying assumptions, we show how supply disruptions at different parts of the network affect inventory decisions. In addition, we quantify the effects of ignoring the disruptions at different parts of the supply chain. Our results provide guidance to company managers by pointing out the disruption characteristics and parts of the supply chains which need to be considered to minimize the effects of the supply disruptions.

3. Disruptions in OWMR Systems with Identical Retailers

In this section we examine the impact of supply disruptions on locally controlled one-warehouse N-retailer systems. Each location monitors only its own inventory level. Retailers observe their demands and place orders with the warehouse. The warehouse observes the orders from each retailer and places its own order with an outside supplier, which is assumed to have infinite supply. Each location decides on its own order level. The inventory levels are reviewed periodically and a base-stock policy is used for replenishment. The leadtimes are zero, which means that if an order is placed at period t and the upstream location has enough inventory to satisfy this order, items arrive at the beginning of period $t+1$. Hence, the inventory/backorder level of the warehouse at period t affects the retailers' inventory/backorder levels at period $t+1$. Demand at each retailer is deterministic. For now, we assume that all retailers are identical and face deterministic demand of d per period. The identical-retailer assumption will be relaxed in Section 4.

A holding cost of h_0 per unit per period is incurred at the warehouse. Similarly, a holding cost of h_r per unit per period is incurred at the retailers. We make no assumptions about the relative magnitude of h_0 and h_r. Unmet demands are backordered and a stockout penalty of p_r per unit per period is incurred at each retailer. Incurred costs are calculated at the end of each period, after shipments.

Disruptions follow a random process governed by probability mass function $\pi_{i,j}$, where i and j are the number of consecutive disrupted periods at the supply processes of the warehouse and the retailers, respectively. We do not restrict the way these probabilities are calculated. In our numerical analysis, we model the disruptions using an infinite-state discrete-time Markov chain, but the following analysis is general enough to handle other types stationary processes. We assume that disruptions are independent over time. For now, we assume that a disruption at the retailers affects all the retailers simultaneously. This is a restrictive assumption and we make it for tractability reasons. However, there are many real-world examples justifying it. For example, if all retailers are under the same labor union, which might strike, then they all will be disrupted together. If all retailers are located close to each other, natural disasters might effect them all. We relax the assumption of simultaneous retailer disruptions in Section 4.

During disruptions in the supply process of the warehouse, the warehouse cannot receive any items from its supplier but it can ship to the retailers as long as it has enough inventory. On the other hand, during disruptions in the supply processes of the retailers, retailers cannot receive the items shipped by the warehouse and these items wait between the warehouse and the retailers and incur holding cost at a rate h_0. In order to simplify the analysis, we assume that disruptions at the warehouse and the retailers never overlap. In addition, each location has enough time to recover, i.e., go back to its non-disrupted inventory level, from the disruption before another one happens. It is possible to provide a Markov chain model which formulates the general problem with overlapping disruptions. However, this formulation requires a state definition with many variables and does not add significantly to the results and insights of the paper.

3.1. Preliminaries

Due to our assumption of non-overlapping disruptions, we have $\pi_{i,j}=0$ if $i,j>0$. We would like to find the optimal base-stock level for the warehouse, s_0^*, and the optimal base-stock level of the retailers, s_r^*. The expected cost of the system for any warehouse and retailer base-stock levels, s_0 and s_r, is the following.

$$\begin{aligned} C(s_0,s_r) =&\pi_{0,0}\left[h_0 s_0 + N\left(h_r\left[s_r-d\right]^+ + p_r\left[d-s_r\right]^+\right)\right] \\ &+\sum_{i=1}^{\infty}\pi_{i,0}\left[h_0\left[s_0-iNd\right]^+ + N\left(h_r\left[s_r-\frac{1}{N}(iNd-s_0)^+ - d\right]^+ + p_r\left[d+\frac{1}{N}(iNd-s_0)^+ - s_r\right]^+\right)\right] \\ &+\sum_{j=1}^{\infty}\pi_{0,j}\left[h_0\left(s_0+jNd\right) + N\left(h_r\left[s_r-(j+1)d\right]^+ + p_r\left[(j+1)d-s_r\right]^+\right)\right] \end{aligned}$$

The first line of the expected cost function is the cost incurred when none of the locations is disrupted times the probability of having no disruptions, i.e., $\pi_{0,0}$. At the beginning of the non-disrupted period, the warehouse's base-stock level is s_0. It ships Nd units to the retailers. These units stay in the pipeline until the beginning of the next period. As stated before, the system incurs a holding cost for these items. Hence, the total holding cost for the items still at the warehouse and the items in the pipeline is $h_0\Big((s_0-Nd)+Nd\Big)$. At the beginning of a non-disrupted period, each retailer has s_r units, and during this period customer demands for d units occur at each. Hence, the second expression inside the brackets in the first line is the total holding and penalty cost charged to the retailers.

The second line is the expected cost when disruptions happen in the supply process of the warehouse only. If positive, s_0-iNd is the on-hand inventory at the warehouse and if negative, it is the backorders at the warehouse at the end of the i^{th} disrupted period. Therefore, $h_0[s_0-iNd]^+$ is the warehouse holding cost at the end of this period. The second expression inside the brackets is the total retailers' cost at the end of the i^{th} disrupted period. $\frac{1}{N}(iNd-s_0)^+$ is the warehouse backorders due to the demands of one of the retailers. The retailers are affected by these backorders in a similar way as retailers in OWMR systems with no disruptions (Zipkin 2000).

The last line is the expected cost for the case where disruptions happen in the supply processes of the retailers only. As stated before, the warehouse is not only charged the holding cost of its own inventory but it is also charged the holding cost of items that cannot be shipped to the retailers due to disruptions at their supply systems. Notice from the second expression inside the brackets that, during disruptions, the retailers can only use their own inventories to satisfy the customer demands. If they have excess inventories they pay holding costs and if they have backorders they pay penalty costs.

Before studying the expected cost function and finding the optimal base-stock levels, with the following theorem, we show that the cost function is piece-wise linear in s_0 and s_r with the breakpoints at integer multiples of Nd and d, respectively.[1]

Theorem 1. *$C(s_0,s_r)$ is a piece-wise linear function in s_0 and s_r with breakpoints at integer multiples of Nd and d, respectively.*

This implies the following result.

[1] All the proofs are in the Appendix.

COROLLARY 1. *The optimal base-stock levels at the warehouse and the retailers are integer multiples of Nd and d, respectively.*

We initially fix the warehouse base-stock level and find the corresponding optimal retailer base-stock levels. Since the retailers are identical their optimal base-stock levels will be the same. From now on $s_r^*(s_0)$ and $C(s_0, s_r^*(s_0))$ represent the optimal retailer base-stock level when the warehouse base-stock level is fixed at s_0 and the optimal expected cost for fixed s_0, respectively. (In the subsequent sections we find the optimal s_0.) The following theorem summarizes the results.

THEOREM 2. *When disruptions happen at the supply processes of both the warehouse and the retailers and $s_0 = kNd$ with $k \geq 0$,*
1. *The cost function $C(s_0, s_r)$ is convex in s_r*
2. *$s_r^*(kNd) = m^*d$, where m^* is the smallest integer $m \geq 1$ to satisfy* (2)

$$\pi_{0,0} + \sum_{i=1}^{m+(k-1)} \pi_{i,0} + \sum_{j=1}^{m-1} \pi_{0,j} \geq \frac{p_r}{p_r + h_r}. \tag{1}$$

Supply chains are globally dispersed and it is possible that only some of the players are subject to supply disruptions. If disruptions only happen at the supply process of the warehouse, we have $\pi_{i,j} = 0 \ \forall j > 0$. Let $\pi_i = \pi_{i,0} \ \forall i \geq 0$ and define $F(i)$ as the cumulative distribution function of π_i. Similarly, if disruptions only happen at the supply processes of the retailers, we have $\pi_{i,j} = 0 \ \forall i > 0$. Let $\pi_j = \pi_{j,0} \ \forall j \geq 0$ and define $G(i)$ as the cumulative distribution function of π_j. Therefore, the warehouse [retailers] is [are] disrupted for precisely i [j] consecutive periods with probability π_i [π_j] and it is [they are] disrupted for i [j] periods or fewer with probability $F(i)$ [$G(j)$]. We define $F^{-1}(x) = \min\{i : F(i) \geq x\}$ as the inverse cumulative distribution function of π_i and similarly foe $G^{-1}(x)$.

Given these definitions, we can state the following Lemma.

LEMMA 1. 1. *If disruptions only happen at the supply process of the warehouse, then $s_r^*(kNd) = m^*d$, where m^* is the smallest integer $m \geq 1$ to satisfy the inequality*

$$F\big(m + k - 1\big) \geq \frac{p_r}{p_r + h_r}. \tag{2}$$

2. *If disruptions only happen at the supply processes of the retailers, then $s_r^*(kNd) = m^*d$, where m^* is the smallest integer $m \geq 1$ to satisfy the inequality*

$$G\big(m - 1\big) \geq \frac{p_r}{p_r + h_r}. \tag{3}$$

According to the first part of this lemma, when the warehouse holds high amounts of inventory, a disruption in the supply system of the warehouse does not affect the retailers and it is enough for each retailer to hold only one period's demand worth of inventory. That is to say, for big k, $s_r^*(s_0) = d$. This claim is supported by (2). For any k that satisfies $F(k-1) \geq \frac{p_r}{p_r + h_r}$, we have $s_r^*(kNd) = d$. According to the second part of the lemma, when disruptions only happen at the supply processes of the retailers, the optimal retailer base-stock levels do not depend on the warehouse base-stock level. This result reflects the fact that during disruptions, the warehouse cannot send any inventories to the retailers, therefore, no matter how much stock it keeps, it does not benefit the retailers.

As a next step, we want to find an expression for the optimal warehouse base-stock level. Having bivariate random variables, $\pi_{i,j}$, prevents us from writing an expression for $s_r^*(s_0)$ in closed form even assuming $\pi_{i,j} = 0$ if $i, j > 0$. This is why we cannot write the cost function $C(s_0, s_r^*(s_0))$ and find the optimal warehouse base-stock level. Hence, we analyze the original expected cost function $C(s_0, s_r)$. We check whether it is convex in s_0 for a fixed value of s_r. For this, we need the following results.

Proposition 1. *For fixed s_r,*

$$\Delta_{s_0} C(s_0,s_r) = Nd\left(h_0\left(\pi_{0,0}+\sum_{j=1}^{\infty}\pi_{0,j}+\sum_{i=1}^{\frac{s_0}{Nd}}\pi_{i,0}\right)+h_r\sum_{i=\frac{s_0}{Nd}+1}^{\frac{s_0}{Nd}+\frac{s_r}{d}-1}\pi_{i,0}-p_r\sum_{i=\frac{s_0}{Nd}+\frac{s_r}{d}}^{\infty}\pi_{i,0}\right) \tag{4}$$

$$\Delta^2_{s_0} C(s_0,s_r) = Nd\left(\left(h_0-h_r\right)\pi_{\frac{s_0}{Nd}+1,0}+\left(h_r+p_r\right)\pi_{\frac{s_0}{Nd}+\frac{s_r}{d},0}\right).$$

Note that for $h_r > h_0$, $\Delta^2_{s_0} C(s_0,s_r)$ is not guaranteed to be non-negative. When s_r is fixed to d, $\Delta^2_{s_0} C(s_0,s_r)$ equals $(h_0+p_r)\pi_{\frac{s_0}{Nd}+1,0}$ and $C(s_0,s_r)$ is convex in s_0. However, for $s_r \geq 2d$ and Markovian disruption processes, we have $\pi_{\frac{s_0}{Nd}+1,0} > \pi_{\frac{s_0}{Nd}+\frac{s_r}{d},0}$ since $\pi_{i,0}$ is decreasing in i. Thus, as h_r increases, the first term inside the brackets decreases faster than the second term. Hence, for $h_r \gg h_0$, $\Delta^2_{s_0} C(s_0,s_r) < 0$. As a result, we cannot claim the convexity of the cost function for $h_r > h_0$. On the other hand, when $h_r \leq h_0$, we have the following result.

Theorem 3. *When $h_r \leq h_0$, $C(s_0,s_r)$ is convex in s_0.*

In Section 3.4 we analyze two separate cases, with $h_r \leq h_0$ and $h_r > h_0$. Before that, we analyze two specific cases, where disruptions happen only at supply process of either the warehouse (Section 3.2) or the retailers (Section 3.3). For these two cases, we can write the cost function $C(s_0, s^*_r(s_0))$ and find the optimal warehouse base-stock levels explicitly.

3.2. Disruptions in the Supply Process of the Warehouse

In this section, we assume that disruptions occur only at the warehouse, that is, $\pi_{i,j} = 0$ for $j > 0$. Theorem 3.2 describes the shape of the cost function $C(s_0, s^*_r(s_0))$.

Theorem 4. *Given that t^* is the smallest integer t to satisfy the inequality $F(t) \geq \frac{p_r}{p_r+h_r}$, the cost function $C(s_0, s^*_r(s_0))$ is*
1. *convex in s_0 for $s_0 \geq t^* Nd$*
2. *non-increasing and concave in s_0 for $s_0 \leq (t^*-1)Nd$, if $h_r > h_0$,*
3. *non-decreasing and convex in s_0 for $s_0 \leq (t^*-1)Nd$, if $h_r \leq h_0$.*

According to this theorem, when $h_r > h_0$, the cost function $C(s_0, s^*_r(s_0))$ is non-increasing and concave up to a point and convex afterward. On the other hand, when $h_r \leq h_0$, the cost function $C(s_0, s^*_r(s_0))$ is non-decreasing and convex. The function $C(s_0, s^*_r(s_0))$ is drawn for both cases in Figure 1. The parameter values we used are $d = 5$, $p_r = 15$, $\alpha_0 = 0.9$ and $\beta_0 = 0.1$ for both parts[2] and $(h_0,h_r) = (1,5)$ and $(h_0,h_r) = (5,1)$ for the first and the second parts of the figure, respectively.

The warehouse base-stock value which minimizes the function $C(s_0, s^*_r(s_0))$ depends on the relative magnitudes of h_0 and h_r. We state the optimal warehouse base-stock level in Theorem 3.2.

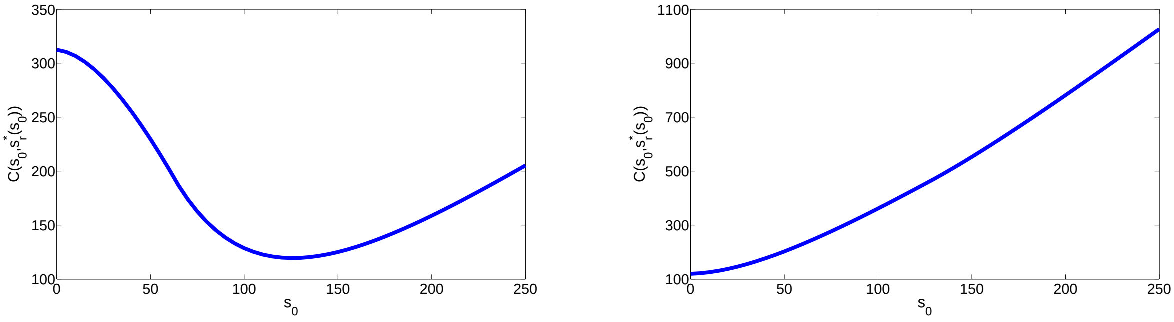
Theorem 5. *The optimal warehouse base-stock level is*
1. *$s^*_0 = k^* Nd$, where k^* is the smallest integer k that satisfies $F(k) \geq \frac{p_r}{h_0+p_r}$, if $h_r > h_0$,*
2. *$s^*_0 = Nd$, if $h_r \leq h_0$.*

According to this theorem, when $h_r > h_0$, we have $s^*_0 = k^* Nd$ with $F(k^*) \geq \frac{p_r}{h_0+p_r}$. Since $h_r > h_0$, it is also true that $F(k^*) \geq \frac{p_r}{h_r+p_r}$. Then Lemma 3.1 implies that $s^*_r(k^* Nd) = d$.

On the other hand, if $h_r \leq h_0$, we have $s^*_0 = Nd$. Lemma 3.1 implies that the corresponding optimal retailer base-stock level is $dF^{-1}\left(\frac{p_r}{p_r+h_r}\right)$. We summarize these results in the first part of Table 1.

These results suggest that the location that stocks the excess inventory to be used as a precaution against disruptions is the one with the smaller holding cost. In fact, we can think of each retailer

[2] Refer to Section 5 for definitions of α_0 and β_0.



(a) (b)

Figure 1 $C(s_0, s_r^*(s_0))$ **vs.** s_0 **for (a)** $h_r > h_0$ **and (b)** $h_r \le h_0$

Warehouse Disrupted	s_0^*	s_r^*
$h_r > h_0$	$NdF^{-1}\left(\frac{p_r}{p_r+h_0}\right)$	d
$h_r \le h_0$	Nd	$dF^{-1}\left(\frac{p_r}{p_r+h_r}\right)$
Retailers Disrupted	s_0^*	s_r^*
$h_r > h_0,\ h_r \le h_0$	0	$d\left(G^{-1}\left(\frac{p_r}{p_r+h_r}\right)+1\right)$

Table 1 **Optimal base-stock levels**

operating as a single-stage system that places orders from a copy of the warehouse and obtain the same results. In order to prove this, we can write the cost function as a sum of N identical functions. Each of these functions is the cost of a single location containing $\frac{1}{N}^{th}$ of the warehouse and a retailer. After solving these single-stage systems, we equate the base-stock level of the warehouse to the sum of the base-stock levels of the portions of the warehouse appearing in all the single-stage systems. Proving the optimality of this approach for OWMR systems with identical retailers is straightforward. Following this logic, we can determine the optimal base-stock level for the whole system using the following theorem, which follows from results by Güllü et al. (1997) or Tomlin (2006); see also Schmitt et al. (2010b) for details.

THEOREM 6. *For N identical retailers with deterministic demands and supply disruptions with cdf $F(i)$,*

1. *the expected cost per period is given by*

$$C(s_r) = N\sum_{i=0}^{\infty}\pi_i\left[h_r\left(s_r-(i+1)d\right)^+ + p_r\left((i+1)d-s_r\right)^+\right]$$

2. *$C(s_r)$ is convex in s_r.*
3. *$s_r^* = m^*d$, where m^* is the smallest integer m such that $F(m-1) \ge \frac{p_r}{p_r+h_r}$.*

3.3. Disruptions in the Supply Processes of the Retailers

According to Lemma 3.1, when disruptions only happen in the supply processes of the retailers, the optimal retailer base-stock levels for a fixed warehouse base-stock level, i.e., $s_r^*(s_0)$, do not depend on s_0. In fact, for this case, the expected cost is an increasing function of s_0. It is easy to see that $s_0^* = 0$. As a result, only the retailers keep inventories of more than one period's demand to prevent customer backorders during disruptions. Note that, unlike the previous case (with disruptions in the supply process of the warehouse) the optimal base-stock levels do not depend on the relative magnitude of the unit holding costs. Even when the warehouse has smaller holding cost, it does not benefit the system to hold inventory there, because these inventories cannot be sent to the retailers during disruptions. Refer to the second part of Table 1 for a summary of the results.

3.4. Disruptions in the Supply Processes of Both the Warehouse and the Retailers

Next, we go back to the case where disruptions happen in the supply processes of both the warehouse and the retailers. For this case, we already know that for $h_r > h_0$, $C(s_0, s_0)$ is not guaranteed to be convex in s_0. However, it is convex in s_0 for $h_r \le h_0$ (Theorem 3.1). We analyze these two cases separately.

3.4.1. Case # 1: $h_r \le h_0$ Convexity allows us to find the optimal warehouse base-stock level for a given retailer base-stock level. The result is summarized in Theorem 3.4.1.

THEOREM 7. *For $h_r \le h_0$, $s_0^*(s_r) = k^* N d$, where k^* is the smallest integer k satisfying the following inequality:*

$$h_0\Big(\pi_{0,0} + \sum_{j=1}^{\infty} \pi_{0,j} + \sum_{i=1}^{k} \pi_{i,0}\Big) + h_r \sum_{i=k+1}^{k+\frac{s_r}{d}-1} \pi_{i,0} - p_r \sum_{i=k+\frac{s_r}{d}}^{\infty} \pi_{i,0} \ge 0 \tag{5}$$

In fact, we can solve expressions (5) and (2) simultaneously and find the optimal base-stock levels since the Hessian matrix, H, is positive semi-definite. This is proven in Theorem 3.4.1.

THEOREM 8. *When $h_r \le h_0$, the Hessian matrix of the function $C(s_0, s_r)$ is positive semi-definite.*

When $h_r \le h_0$, it is expensive for the warehouse to hold inventory. Note that $s_0 = 0$ satisfies inequality (5), given that s_r satisfies (2). Hence, no matter what the disruption probabilities are, $s_0^* = 0$ and $s_r^* = m^* d$, where m^* is the smallest integer m to satisfy (2).

3.4.2. Case # 2: $h_r > h_0$ When the unit holding cost of the warehouse is smaller than that of the retailers, $C(s_0, s_r)$ is not guaranteed to be convex. Solving for the optimal base-stock levels requires complete enumeration (i.e., using the Projection Algorithm (Zipkin 2000)). However, this can be quite inefficient. Here, we suggest an easy-to-implement heuristic procedure.

For OWMR systems with stationary and random demand and no disruptions, a retailer's base-stock level is a decreasing function of its local unit holding cost. In addition, since it is more expensive for the whole system to hold inventory, the total of the base-stock levels of the system decreases with a retailer's local unit holding cost. Given the optimal base-stock levels for $h_r = h_0$, suppose that we increase h_r by a small amount. We expect one of the following three changes to happen:

1. An increase in the warehouse base-stock level by Nd and a decrease in the retailer base-stock level by d (total stays the same),
2. A decrease in the retailer base-stock level by d (total decreases by Nd) or
3. A decrease in the warehouse base-stock level by Nd (total decreases by Nd).

Accordingly, we assume that the total base-stock level will not change by more than Nd when h_r changes by a small amount. Based on this assumption, we propose the following heuristic for approximating the optimal warehouse and retailer base-stock levels when $h_r > h_0$:

HEURISTIC 1.

1. Assume $h_r = h_0$ and calculate s_0^* and s_r^* using Table 1. Set $s_0 = s_0^*$ and $s_r = s_r^*$.

2. Solve the following three expressions and obtain the values of h_r^1, h_r^2 and h_r^3.

$$C(s_0,s_r)-C(s_0+Nd,s_r-d) = 0$$

$$h_r^1 = \frac{h_0\left(1-\sum_{i=\frac{s_0}{Nd}+1}^{\infty}\pi_{i,0}\right)+p_r\left(2-\frac{s_r}{d}\right)^+\sum_{i=1}^{\frac{s_0}{Nd}}\pi_{i,0}+p_r\sum_{j=\frac{s_r}{d}-1}^{\infty}\pi_{0,j}}{\left(\frac{s_r}{d}-1-\left(\frac{s_r}{d}-2\right)^+\right)\left(\pi_{0,0}+\sum_{i=1}^{\frac{s_0}{Nd}}\pi_{i,0}\right)+\sum_{j=1}^{\frac{s_r}{d}-2}\pi_{0,j}}$$

$$C(s_0,s_r)-C(s_0,s_r-d) = 0$$

$$h_r^2 = \frac{p_r-p_r\left(\pi_{0,0}+\sum_{i=1}^{\frac{s_0}{Nd}+\frac{s_r}{d}-2}\pi_{i,0}+\sum_{j=1}^{\frac{s_r}{d}-2}\pi_{0,j}\right)}{\left(\pi_{0,0}+\sum_{i=1}^{\frac{s_0}{Nd}+\frac{s_r}{d}-2}\pi_{i,0}+\sum_{j=1}^{\frac{s_r}{d}-2}\pi_{0,j}\right)}$$

$$C(s_0,s_r)-C(s_0-Nd,s_r) = 0$$

$$h_r^3 = \frac{p_r\sum_{i=\frac{s_0}{Nd}+\frac{s_r}{d}-1}^{\infty}\pi_{i,0}-h_0\left(\pi_{0,0}+\sum_{i=1}^{\frac{s_0}{Nd}-1}\pi_{i,0}+\sum_{j=1}^{\infty}\pi_{0,j}\right)}{\sum_{i=\frac{s_r}{d}}^{\frac{s_0}{Nd}+\frac{s_r}{d}-2}\pi_{i,0}}$$

Let $h_r^{bound} = \texttt{min}\{h_r^1, h_r^2, h_r^3\}$.

3. If $h_r^{bound} = h_r^1$, set $s_0 = s_0 + Nd$ and $s_r = s_r - d$; else if $h_r^{bound} = h_r^2$, set $s_0 = s_0$ and $s_r = s_r - d$; else if $h_r^{bound} = h_r^3$, set $s_0 = s_0 - Nd$ and $s_r = s_r$.

4. $\forall h_r \in (h_0, h_r^{bound}]$, set the warehouse base-stock level to s_0 and the retailers' base-stock levels to s_r. If our original h_r is in this interval, stop. Else continue with step 2, until obtaining a bound that includes the original unit retailer holding cost.

In the first step of the heuristic, we equate h_r to h_0 and solve for the base-stock levels. Via the second step, we initially find the smallest retailer unit holding cost greater than h_0 such that one of the changes described above happens. For example, if h_r is increased from h_0 to h_r^1, s_0 will increase to $s_0 + Nd$ and s_r will decrease to $s_r - d$. Similarly, h_r^2 and h_r^3 correspond to the smallest retailer holding costs which enforce changes #2 and #3, respectively. In the third step of the algorithm, we equate h_r to $\texttt{min}\{h_r^1, h_r^2, h_r^3\}$ and make the change which corresponds to the minimum. We continue increasing the retailer unit holding cost until h_r is obtained.

Note that this heuristic is based on the assumption that with gradual increases in the unit holding cost of the retailers, the total base-stock level of the system decreases at most by Nd. Hence, it will not detect any sudden jumps in the base-stock levels. Actually, sudden jumps appear to be optimal rarely (or never): in all of the numerical analysis we have performed, we have not encountered any case where the heuristic does not give the optimal results.

One of the advantages of this heuristic is that, by solving a single problem instance with warehouse and retailer unit holding costs of h_0 and h_r, respectively, all the other problems with retailer unit holding costs smaller than h_r are automatically solved. Indeed, this heuristic shows the sensitivity of the optimal base-stock levels to changes in the holding costs.

4. Disruptions in OWMR Systems with Non-identical Retailers

In this section, we relax the assumption of identical retailers and allow different per-period demands, unit holding and backordering costs at the retailers. Moreover, we assume that retailers are subject to different and independent disruption processes. Hence, the random variable π has $N+1$ variables, one for the warehouse and one for each retailer. We retain our assumption of non-overlapping disruptions at the warehouse and the retailers due to tractability reasons. By allowing different disruption processes at the retailers, we capture the fact that some retailers might be disrupted more often than others and some might have the ability to recover faster than others. Table 2 summarizes the notation used throughout the section.

Variable	Description
0	index for the warehouse
r	index for the retailers, $r \in \{1, 2, .., N\}$
s_0	warehouse base-stock level
s_r	base-stock level of retailer r
s	vector of retailer base-stock levels, $s = \{s_1, s_2, ..., s_N\}$
d_r	per-period demand at retailer r, $D = \sum_{r=1}^N d_r$
h_0	warehouse holding cost per item per period
h_r	unit holding cost per item per period at retailer r
p_r	unit backorder cost per item per period at retailer r
j_r	# of consecutive disrupted periods at retailer r

Table 2 Notation / non-identical retailers

The expected cost function is as follows:

$$\begin{aligned} C(s_0, s) = &\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_N=0}^{\infty} \pi_{0,j_1,j_2,\ldots,j_N}\left[h_0\left(s_0 + \sum_{r=1}^N j_r d_r\right) + \sum_{r=1}^N\left(h_r\left[s_r - (j_r+1)d_r\right]^+ + p_r\left[(j_r+1)d_r - s_r\right]^+\right)\right] \\ &+ \sum_{i=1}^{\infty}\pi_{i,0,0,\ldots,0}\left[h_0\left[s_0 - iD\right]^+ + \sum_{r=1}^N\left(h_r\left[s_r - \frac{d_r}{D}(iD - s_0)^+ - d_r\right]^+ + p_r\left[d_r + \frac{d_r}{D}(iD - s_0)^+ - s_r\right]^+\right)\right] \end{aligned}$$

The first line of this function is the cost incurred when there is no disruption at the warehouse and some disruptions or no disruptions at the retailers. The second line is the expected cost when disruptions only happen in the supply process of the warehouse.

We obtain very similar results as in Section 3 when we fix the warehouse base-stock level to $s_0 = kD$ and find expressions for the corresponding optimal retailer base-stock levels $s_r^*(s_0)$ $\forall r \in \{1, 2, ..., N\}$. In fact, it can be shown that the expected cost is a convex function of s_r $\forall r \in \{1, 2, ..., N\}$. The optimal base-stock level for retailer r is $s_r^*(s_0) = m_r^* d_r$, where m_r^* is the smallest integer $m_r \geq 1$ to satisfy the inequality

$$\sum_{i=1}^{m_r + \frac{s_0}{D} - 1} \pi_{i,0,0,\ldots,0} + \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_r=0}^{m_r-1}\cdots\sum_{j_N=0}^{\infty}\pi_{0,j_1,j_2,\ldots,j_r,\ldots,j_N} \geq \frac{p_r}{p_r + h_r}. \tag{6}$$

Next, we check whether $C(s_0, s)$ is convex in s_0 for fixed value of s. With simple algebra, we can show that

$$\begin{aligned} \Delta_{s_0} C(s_0, s) &= h_0 D\left(\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_N=0}^{\infty}\pi_{0,j_1,j_2,\ldots,j_N} + \sum_{i=1}^{\frac{s_0}{D}}\pi_{i,0,0,\ldots,0}\right) + \sum_{r=1}^N h_r d_r \sum_{i=\frac{s_0}{D}+1}^{\frac{s_0}{D}+\frac{s_r}{d_r}-1}\pi_{i,0,0,\ldots,0} - \sum_{r=1}^N p_r d_r \sum_{i=\frac{s_0}{D}+\frac{s_r}{d_r}}^{\infty}\pi_{i,0,0,\ldots,0} \\ \Delta_{s_0}^2 C(s_0, s) &= \left(h_0 D - \sum_{r=1}^N h_r d_r\right)\pi_{\frac{s_0}{D}+1,0,0,\ldots,0} + \sum_{r=1}^N\left(h_r + p_r\right)d_r \pi_{\frac{s_0}{D}+\frac{s_r}{d_r},0,0,\ldots,0}. \end{aligned}$$

Note that the cost function is not guaranteed to be convex in s_0 for fixed values of the retailer base-stock levels. It is convex when we have $h_r \leq h_0$ $\forall r \in \{1, 2, ..., N\}$ and when $s_r = d$. However, for $s_r \geq 2d$, if $h_0 D - \sum_{r=1}^N h_r d_r < 0$, we cannot claim convexity.

In Section 3.4, for the case with $h_r > h_0$, we proposed a heuristic procedure to approximate the retailer and warehouse base-stock levels. We cannot apply the same heuristic procedure when the retailers have different unit holding costs. Instead, we propose other heuristics. As we do for the identical-retailers case, we initially analyze the cases where disruptions only happen in the supply process(es) of either the warehouse or the retailers.

4.1. Disruptions in the Supply Process of the Warehouse

Let $\pi_i = \pi_{i,0,0,\ldots,0}$ $\forall i \geq 0$ and, similar to the identical-retailers case, define $F(i)$ as the cumulative distribution function of π_i. Therefore, for each $r \in \{1, 2, ..., N\}$, the optimal base-stock level for a fixed warehouse base-stock level is $s_r^*(s_0 = kD) = m_r^* d_r$, where m_r^* is the smallest integer m_r to satisfy the inequality $F(m_r + k - 1) \geq \frac{p_r}{p_r + h_r}$.

In order to determine the optimal warehouse base-stock level, we analyze the function $C(s_0, s^*(s_0))$. Initially, we define

$$u_r^* = \min\left\{u_r : u_r \in \mathbb{Z}, u_r \geq F^{-1}\left(\frac{p_r}{p_r + h_r}\right)\right\}.$$

Based on this definition, $s_r^*(kD)$ can be expressed as $\max\{d_r, (u_r^* - k + 1)d_r\}$. Next, we differentiate between the retailers with $u_r^* \leq k$ and the retailers with $u_r^* \geq k + 1$. Suppose that $u_r^* \leq k$ for $r \in \{1, 2, ..., n\}$ and $u_r^* \geq k + 1$ for $r \in \{n+1, n+2, ..., N\}$. Then, assuming $s_0 = kD$,

$$\begin{aligned}
C(s_0, s^*(s_0)) &= h_0 D \sum_{i=1}^{k} \pi_i (k-i) + \sum_{r=1}^{n} p_r d_r \sum_{i=k+1}^{\infty} \pi_i (i-k) \\
&\quad + \sum_{r=n+1}^{N} \left[h_r d_r \left((u_r^* - k) F(k) + \sum_{i=k+1}^{\infty} \pi - i (u_r^* - i)^+ \right) + p_r d_r \sum_{i=k+1}^{\infty} \pi_i (i - u_r^*)^+ \right] \\
\Delta_{s_0} C(s_0, s^*(s_0)) &= F(k) \left(h_0 D + \sum_{r=1}^{n} p_r d_r - \sum_{r=n+1}^{N} h_r d_r \right) - \sum_{r=1}^{n} p_r d_r \\
\Delta_{s_0}^2 C(s_0, s^*(s_0)) &= \pi_{k+1} \left(h_0 D + \sum_{r=1}^{n} p_r d_r - \sum_{r=n+1}^{N} h_r d_r \right) \\
&= \pi_{k+1} \left(\sum_{r=1}^{n} (h_0 + p_r) d_r + \sum_{r=n+1}^{N} (h_0 - h_r) d_r \right).
\end{aligned}$$

The expression for $\Delta_{s_0}^2 C(s_0, s^*(s_0))$ suggests that unless we have $h_0 \geq h_r$ for all the retailers in $\{n+1, n+2, ..., N\}$, the function $C(s_0, s^*(s_0))$ is not guaranteed to be convex in s_0. Hence, we suggest a heuristic procedure to approximate the optimal warehouse base-stock level. Indeed, this heuristic also approximates the retailer base-stock levels. Let s_0^a be the approximate warehouse base-stock level and $\bar{s}_r^a$ be the vector of approximate retailer base-stock levels suggested by the heuristic.

HEURISTIC 2.

1. Find the smallest integer k to satisfy the following inequality (6) and set the warehouse base-stock level to $s_0^a = kD$:

$$\left(h_0 D + \sum_{r=1}^{n} p_r d_r - \sum_{r=n+1}^{N} h_r d_r \right) F(k) \geq \sum_{r=1}^{n} p_r d_r, \tag{7}$$

where $\{1, 2, ..., n\}$ is the set of retailers with $h_r > h_0$ and $\{n+1, n+2, ..., N\}$ is the set of retailers with $h_r \leq h_0$.

2. For the retailers with $h_r > h_0$, set $s_r^a = d_r$.

3. For the retailers with $h_r \leq h_0$, set $s_r^a = m_r d_r$, where m_r is the smallest integer to satisfy the inequality $F(m_r + k - 1) \geq \frac{p_r}{p_r + h_r}$.

In Section 3.2, we concluded that when $h_r > h_0$, the optimal retailers' base-stock level equals their per-period demand. When $h_r \leq h_0$, the warehouse base-stock level is the same as its per-period demand and the retailers hold more inventory to protect against disruptions. Heuristic 2 makes use of the same logic. In fact, once we have $s_r^a = d_r$ for retailers with $h_r > h_0$, these retailers belong to the group of retailers with $r \in \{1, 2, ..., n\}$. On the other hand, the retailers with $h_r \leq h_0$ belong to the group of retailers with $r \in \{n+1, n+2, ..., N\}$. Note that, by using these retailer base-stock levels, we ensure the non-negativity of $\sum_{r=n+1}^{N} (h_0 - h_r) d_r$ and, therefore, $\Delta_{s_0}^2 C(s_0, s^*(s_0))$. As a result, we approximate the cost function $C(s_0, s^*(s_0))$ with a convex function. This is why we choose to set the warehouse base-stock level by solving the inequality $\Delta_{s_0} C(s_0, s^*(s_0)) \geq 0$, i.e., inequality (6).

We elaborate on the performance of this heuristic approach in Section 5.1.3.

4.2. Disruptions in the Supply Processes of the Retailers

In this section, we assume that the supply process of the warehouse is perfectly reliable but the supply processes of the retailers are subject to disruptions. Given the independence of the disruption processes among the retailers, we can define π_j^r and $G_r(j)$ as the pmf and cdf of the disruption process at retailer r.

The expected cost for any warehouse base-stock level s_0 and retailer base-stock levels s is as follows:

$$C(s_0, s) = h_0 s_0 + \sum_{r=1}^{N} \sum_{j=1}^{\infty} \pi_j^r \left(h_0 j d_r + h_r \Big[s_r - (j+1) d_r \Big]^+ + p_r \Big[(j+1) d_r - s_r \Big]^+ \right)$$

Following the same steps as in Section 3.3, we can solve the system to optimality and we can conclude the optimal results given in Table 3.

	s_0^*	s_r^*
$h_r > h_0$, $h_r \le h_0$	0	$d_r \left(G_r^{-1} \left(\frac{p_r}{p_r + h_r} \right) + 1 \right)$

Table 3 Disruptions at the retailers

4.3. Disruptions in the Supply Processes of Both the Warehouse and the Retailers

Next, we return to the case where disruptions happen in both the warehouse's and the retailers' supply processes. For this case, we propose another heuristic approach to approximate the base-stock levels of all the locations. Initially, we decompose the OWMR system into N serial systems and retain all the parameters. We define s_{0r} to be the base-stock level of the warehouse belonging to the serial system with retailer r. For each serial system we use our results from Section 3.4 (applied to these serial systems as a special case) to obtain the base-stock levels of the serial systems. Finally, we aggregate the serial systems back into the OWMR system and sum the base-stock levels of the copies of the warehouse in the serial systems to approximate the warehouse base-stock level. This approach is similar in spirit to the Decomposition-Aggregation heuristic proposed by Rong et al. (2012) for distribution systems with demand uncertainty and no disruptions. In contrast to our heuristic, the Decomposition-Aggregation heuristic uses *backorder matching* to determine the warehouse base-stock level.

Let s_0^b be the approximate warehouse base-stock level and s^b be the vector of approximate retailer base-stock levels suggested by the heuristic.

HEURISTIC 3.

1. Decompose the OWMR system into N serial systems, each consisting of a copy of the warehouse and a single retailer, and retain all the system parameters.
2. For each $r \in \{1, 2, ..., n\}$ (where n is as defined in Section 4.1), apply Heuristic 1 to determine s_r^b and s_{0r}^b.
3. For each $r \in \{n+1, n+2, ..., N\}$, set $s_{0r}^b = d_r$ and find m^*, which is the smallest integer m to satisfy inequality (2) with $k = 1$. Set $s_r^b = m^* d_r$.
4. Set $s_0^b = \sum_{r=1}^N s_{0r}^b$.

We elaborate on the performance of this heuristic in Section 5.1.3.

5. Numerical Results

In the previous section, we analyzed two cases (identical and non-identical retailers) each with three scenarios (based on where disruptions occur). Although we can solve to optimality 3 out of the 6 scenarios, we proposed heuristics for the rest. In this section, we evaluate the performances of these heuristics and elaborate on the effect of disruptions on optimal inventory decisions. In addition, we provide some managerial insights.

When disruptions occur in the supply process(es) of either the warehouse or the retailers, we assume that the disruptions follow a random process governed by a univariate random variable π_k ($k=i$ for the warehouse, $k=j$ for the retailers). In order to obtain these probabilities, we model the disruptions using an infinite-horizon discrete-time Markov Chain (DTMC). The state is defined to be the number of consecutive periods that have been disrupted. We define α and β to be the disruption and recovery probabilities, respectively. Hence, given that a location is in the non-disrupted state in a period, it is disrupted in the next period with probability α and given that it is disrupted in a period, it recovers in the next period with probability β. Based on these definitions, the long-run distribution of being in state k, i.e., π_k values, are given as $\pi_0 = \frac{\beta}{\alpha+\beta}$ and $\pi_k = \frac{\alpha\beta}{\alpha+\beta}(1-\beta)^{k-1}\ \forall k \geq 1$ (Snyder and Shen 2011).

On the other hand, when disruptions happen in the supply processes of both the warehouse and the retailers, we model the disruptions using an infinite-horizon discrete-time Markov chain, where the state, (i,j), has two components: i represents the number of consecutive disrupted periods in the warehouse and j represents the number of consecutive disrupted periods in the supply processes of the retailers. The state space is $\{(i,j): i=0 \text{ or } j=0\}$. We define α_0 (β_0) and α_r (β_r) as the disruption (recovery) probabilities at the warehouse and the retailers, respectively. Our assumption of non-overlapping disruptions at the warehouse and retailers requires the constraint $\alpha_0 + \alpha_r \leq 1$ to be satisfied. Based on these definitions the steady state probabilities can be determined as in Lemma 5.

Lemma 2. *For a OWMR system with the warehouse and the retailers' supply processes subject to disruptions with disruption and recovery probabilities (α_0, β_0) and (α_r, β_r), respectively, the steady state probabilities are*

$$\begin{aligned}
\pi_{0,0} &= \frac{\beta_0\beta_r}{\beta_0\beta_r + \alpha_0\beta_r + \alpha_r\beta_0} \\
\pi_{i,0} &= \alpha_0(1-\beta_0)^{i-1}\pi_{0,0}, \qquad i \geq 1 \\
\pi_{0,j} &= \alpha_r(1-\beta_r)^{j-1}\pi_{0,0}, \qquad j \geq 1.
\end{aligned}$$

5.1. Performance of the Heuristics

5.1.1. Performance of Heuristic 1 In order to assess the performance of Heuristic 1, we apply it to problems with 18,225 different parameter combinations. We consider 3 retailers and fix d at 5. We fix h_0 at 3 and p_r at 10. h_r is drawn from a uniform distribution on $[3,15]$. (Recall that the problem can be solved without the heuristic if $h_r \leq h_0$.) We also vary the disruption and recovery probabilities (α_0, β_0 and β_r) between 0.1 and 0.9 with 0.1 increments. For a given α_0, we vary α_r from $1-\alpha_0$ to 0.9.[3] For each probability combination, we tested 5 h_r values.

We calculate the exact values of the base-stock levels by enumeration. In all 18,225 of the instances, the heuristic procedure gives the exact result in negligible time. Hence, not only does the heuristic provide a simple procedure to calculate the base-stock levels but it also has an average percentage error of 0 for our instances. This implies that our assumption that a very small increment in the retailers' unit holding cost results in no more than one period's worth of demand change in

[3] Due to the constraint $\alpha_0 + \alpha_r \leq 1$, we have 45 possible (α_0, α_r) combinations. Together with 81 combinations for (β_0, β_r), the total number of combinations for the disruption and recovery probabilities is 3,645.

the base-stock levels of both the warehouse and the retailers is indeed reasonable. Note that the way we generate the disruption probabilities makes $\pi_{i,j}$ monotonically decreasing in i and j. This is one of the reasons for not having jumps bigger than the per-period demand in the base-stock values with small changes in h_r. Other, non-Markovian, ways of generating these probabilities, like assuming directly $\pi_{0,0} = \pi_{1,0} = \pi_{2,0} = 0.3$ and $\pi_{0,1} = \pi_{0,2} = 0.05$, may make our assumption invalid.

5.1.2. Performance of Heuristic 2 We examine the performance of Heuristic 2 by generating 14,580 random data sets. We fix N at 3 and vary $[d_1, d_2, d_3]$, h_0, h_r, p_r, α and β. For each combination of α, β, h_0 and $[d_1, d_2, d_3]$, we randomly generate 20 (h_r, p_r) pairs for each retailer. The values tested for these parameters are given in Table 4.

Variable	Values
α, β	$\{0.1, 0.2, ..., 0.9\}$
h_0	5; 10; 15
h_r	$\sim U[1, 20]$
p_r	$\sim U[2, 30]$
$[d_1, d_2, d_3]$	[1,1,1]; [2,5,3]; [10,1,10]

Table 4 Parameter values for the numerical experiment

We calculate the exact values of the base-stock levels by enumeration and the percentage errors by $\%\epsilon = \frac{C(s_0, s_r) - C(s_0^*, s_r^*)}{C(s_0^*, s_r^*)} 100$. Here s_0 and s_r are the warehouse and retailers' base-stock levels suggested by the heuristic.

In general, the heuristic performs extremely well. The average percentage difference between the optimal cost and the cost suggested by the heuristic is 0.60%. The standard deviation of the percentage error, σ_ϵ, is 2.22%. In 87.96% of the instances, the heuristic gives the optimal result. The average percentage error for the other 12.04% of the cases is 4.96%. Average CPU times for the enumeration procedure and the heuristic are 5.54 and 9.73×10^{-4}, respectively.

We observe that in the cases where the heuristic does not provide the optimal solution, it overestimates the true warehouse base-stock level while it underestimates the true retailer base-stock levels. This result is expected since the heuristic does not take into account the possible interactions among the retailers' base-stock levels. Once we set a retailer's base-stock level to its per-period demand, it implies keeping more inventory at the warehouse to protect this retailer against supply disruptions. In fact, these warehouse inventories might result in lower base-stock levels at the other retailers as well.

5.1.3. Performance of Heuristic 3 In order to test the performance of Heuristic 3, we fix N at 2 and vary $[d_1, d_2]$, h_0, h_r, p_r, α_0, β_0, α_r and β_r. For each combination of α_0, β_0, α_r, β_r, h_0 and $[d_1, d_2]$, we randomly generate 5 (h_r, p_r) pairs for each of the retailer. The values tested for these parameters are given in Table 5.

Variable	Values
$\alpha_0, \beta_0, \alpha_r, \beta_r$	$\{0.1, 0.2, ..., 0.9\}$
h_0	5; 10; 15
h_r	$\sim U[1, 20]$
p_r	$\sim U[2, 30]$
$[d_1, d_2]$	[1,1]; [2,5]

Table 5 Parameter values for the numerical experiment

In total, we test 109,350 parameter combinations. The average percentage difference between the optimal cost and the cost suggested by the heuristic is 0.68%. The standard deviation of the percentage error, σ_ϵ, is 2.73%. In 86.46% of the cases the heuristic gives the optimal result and the average percentage error of the other 13.54% of the cases is 5.04%.

Calculation of exact base-stock levels relies on enumeration. Average CPU times for the enumeration procedure and the heuristic are 8.08 and 4.30×10^{-3}, respectively. The computation time increases considerably as we increase the number of retailers. This is why we perform fewer experiments with $N = 3$ and $N = 4$. The heuristic finds the optimal results in more than 90.00% of these cases and the average percentage error is smaller than 0.5%.

For disruption-free OWMR systems with constant leadtimes, random demands and the cost structure like the one studied in this paper, the sum of the base-stock levels of the parts of the warehouse in all the of serial systems constitutes an upper bound on the optimal base-stock level of the warehouse (Rong et al. 2012). The reason is so called "risk-pooling effect", which suggests that the base-stock level of the original system should be smaller that the sum of the base-stock levels of the parts of the warehouse, because high demand at one retailer can be compensated by low demand at another retailer. This result is not valid for the system we study in this paper. The sum $\sum_{r=1}^{N} s_{0r}^b$ is not an upper bound for s_0^*. In fact, it can be smaller or larger than s_0^*. Overall, $\sum_{r=1}^{N} s_{0r}^b$ seems to be a good approximation for s_0^*.

5.2. Comparative Statics

In order to see how the base-stock levels and the corresponding expected costs depend on the disruption parameters, we test $9^4 = 6561$ different combinations of $\alpha_0, \beta_0, \alpha_r$ and β_r. We vary α_0 and α_r from 0.1 to 0.5 with 0.05 increments and β_0 and β_r from 0.1 to 0.9 with 0.1 increments.[4] We assume that there are 3 identical retailers with $h_r = 5$, $p_r = 10$ and $d_r = 5$. For the warehouse, we have either $h_0 = 3$ or $h_0 = 8$. We use Heuristic 1 to calculate the optimal base-stock levels and the expected costs. We take the average of the optimal base-stock levels and the average of the optimal costs and associate these with the particular disruption parameter value. For example, if $s_0^{\alpha_0=0.3}$ represents the average warehouse base-stock level when $\alpha_0 = 0.3$, we calculate it by

$$s_0^{\alpha_0=0.3} = \sum_{i=0}^{9}\sum_{j=0}^{9}\sum_{k=0}^{9} s_0^{\alpha_0=0.3,\beta_0=0.5+0.05i,\alpha_r=0.05+0.05j,\beta_r=0.5+0.05k}/729.$$

Figure 5.2 shows how disruption parameters affect the optimal warehouse base-stock level. The y-axis represents the percentage change given that the base disruption parameter value is its minimum value. For example, the percentage change for any $\alpha_0 = i$ is calculated by

$$\%\text{change} = \frac{s_0^{\alpha_0=i} - s_0^{\alpha_0=0.1}}{s_0^{\alpha_0=0.1}} 100.$$

The same applies to other disruption parameters in the subsequent two graphs for s_r and $C(s_0, s_r)$.

Figure 5.2 suggests that the optimal warehouse base-stock level increases with α_0 and β_r. It is not affected by β_0 or α_r. When $h_r > h_0$, it is favorable to keep inventory at the warehouse. However, the warehouse does not want to keep inventory if it cannot ship to the retailers. As β_r increases, the link between the warehouse and the retailers becomes safer and s_0 increases. As expected, when α_0 increases, the warehouse keeps more inventory to protect against disruptions. We do not include a figure for the case with $h_0 = 8$, since, as suggested by Table 1, independent of the disruption parameters, we have $s_0^* = 15$.

[4] We choose to restrict the values for disruption probabilities due to the constraint $\alpha_0 + \alpha_r \leq 1$. If, for example, $\alpha_0 = 0.9$, this constraint says that the only possible value for α_r is 0.1. This might result in lower average costs than the case where $\alpha_0 = 0.8$ and α_r is either 0.1 or 0.2. However, the average is cost is expected to increase as α_0 increases.

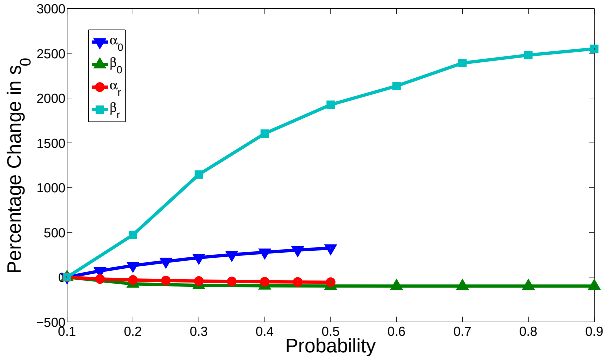
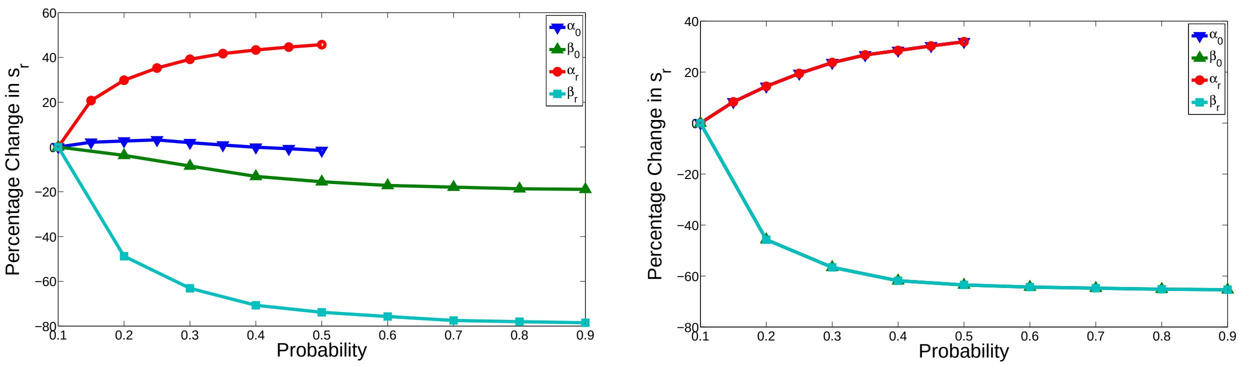


Figure 2 Effect of disruption parameters on the optimal warehouse base-stock level, $h_0 = 3$



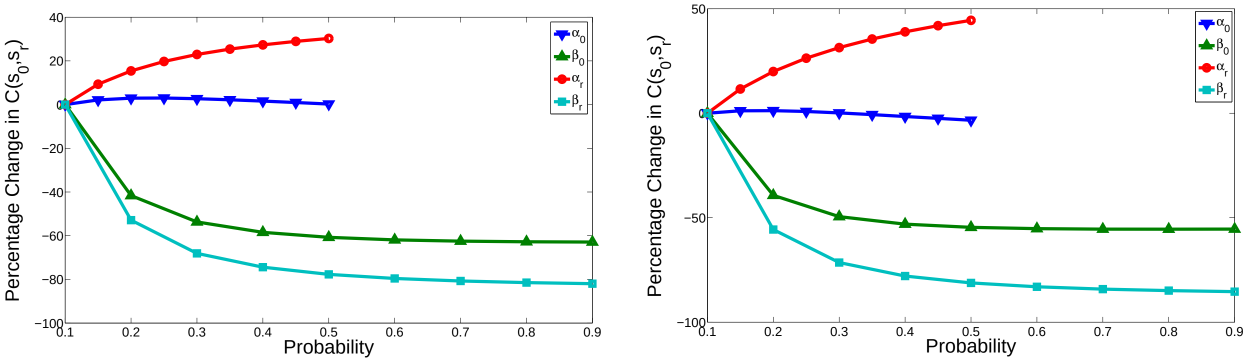
(a) (b)

Figure 3 Effect of disruption parameters on the optimal retailer base-stock levels, (a) $h_0 = 3$**, (b)** $h_0 = 8$

Figure 3.4.1 depicts the effect of disruption parameters on the optimal retailer base-stock levels for two values of h_0. When h_0 is small, it is better to keep inventories at the warehouse. However, with increasing α_r, it becomes unlikely that the warehouse can ship the items it keeps to the retailers. Hence, as suggested by the figure, the retailers need to keep more inventory. Fast recoveries from disruptions in the supply processes of the retailers result in lower base-stock levels at the retailers. Note that, when $h_0 = 3$, α_0 and β_0 do not have a prominent effect on s_r^*. In fact, α_0 causes an increase in s_0^* (Figure 5.2). Hence, it is up to the warehouse to keep more inventory to protect the whole system against the increased disruption probabilities in its supply system.

On the other hand, when $h_0 = 8$, in terms of cost, it is beneficial to keep inventories at the retailers and almost no stock at the warehouse. The s_r^* values are affected in the same manner by the upstream disruptions. Whether disruptions happen in the supply process of the warehouse or in the supply processes of the retailers, s_r^* increases with the disruption probability and decreases with the recovery probability.

Figure 3.4.1 shows the effects of disruption parameters on the optimal expected cost for two values of h_0. The expected cost increases with α_0 and α_r and decreases with β_0 and β_r. For both h_0 values, the effects of the retailers' disruption parameters dominate the effects of the warehouse's disruption parameters. In addition, the recovery probabilities play a more important role than the disruption probabilities. These observations motivate the following section on the managerial insights.



(a) (b)

Figure 4 Effect of disruption parameters on the optimal expected cost, (a) $h_0 = 3$**, (b)** $h_0 = 8$

5.3. Managerial Insights

Real-world supply chains are much more complicated than the system studied in this paper. A company might have a supply chain with many echelons and many players in each echelon. When inventory systems consist of multiple locations with each location's supply system subject to disruptions, it might be very costly to keep more inventory in each location for events which might never happen. Hence, managers of such companies might object to the conclusion that all players of the supply chain should protect against the disruption possibilities in all parts of the chain. Therefore, they face a problem of determining which disruptions they should protect against and which disruptions they can completely ignore. If managers decide to ignore the disruptions at some parts of the supply chain, what would be the consequences? Ignoring the disruptions at which parts of the supply chain is more costly? In this section, we will address these questions using the results we obtain for a simpler (OWMR) system.

Although some disruptions are unavoidable, supply disruptions caused by transportation problems, labor strikes, etc., can be avoided or their probability of occurrence can be lowered. However, as in efforts to reduce the effects of disruptions, reducing their probability of occurrence can be costly. Hence, company managers may want to know whether to invest more on trying to avoid the disruptions, i.e., decreasing the disruption probabilities, or making their systems more resilient against disruptions, i.e., increasing the recovery probabilities. We answer this question in this section as well.

5.3.1. Ignoring the Disruptions We consider a system with one warehouse and two retailers with all the locations subject to disruptions. We have $d_r = 5, h_r = 5, p_r = 10$ for $r \in \{1,2\}$ and $h_0 = 3$ or $h_0 = 8$. We vary disruption and recovery probabilities from 0.1 to 0.9 with 0.1 increments. Hence, the result for each particular h_0 value is based on 3,645 parameter combinations.

Note that, when the system is disruption safe, the optimal base-stock levels are $s_0^* = 0$ and $s_r^* = d_r$. If one location ignores all the disruptions, we equate the base-stock level of that location to its optimal disruption-safe base-stock level and calculate the cost of the system by keeping the base-stock levels of the other locations at their true optimal values. If all locations ignore the disruptions at a particular location of the system, we calculate the optimal base-stock levels with no disruptions at that location. Then, we calculate the cost of ignoring the disruptions.

We analyze 7 different cases and, in Table 6, we summarize the averages of the percentage cost increases resulting from ignoring the disruptions.

According to the results in Table 6, for both $h_0 = 3$ and $h_0 = 8$, the retailers' ignorance is more costly than the warehouse's ignorance. In fact, even if retailers ignore only one type of disruption (either in the supply system of the warehouse or in the supply systems of the retailers), the system

Case	$h_0 = 3$	$h_0 = 8$
Warehouse ignores all the disruptions	2.34	0.00
Retailers ignore the disruptions at the supply system of the warehouse	3.80	11.17
Retailers ignore the disruptions at their supply systems	10.49	9.50
Retailers ignore all the disruptions	22.45	26.68
All locations ignore the disruptions at the supply system of the warehouse	20.31	15.14
All locations ignore the disruptions at the supply systems of the retailers	24.67	9.51
All locations ignore all the disruptions	42.36	31.22

Table 6 Percentage cost increases resulting from ignoring the disruptions

incurs a higher cost than the case where the warehouse ignores all the disruptions. The reason is that the retailers' inventories are used as direct protection against customer backorders. If they ignore the disruptions and keep less inventories than required, the system incurs high backorder costs. On the other hand, if the warehouse ignores any of the disruptions, the retailers can compensate its ignorance and save the system from incurring high backorder costs.

It costs more if the retailers ignore the disruptions in the supply system of the warehouse when $h_0 = 8$ (3.80% for $h_0 = 3$ and 11.17% for $h_0 = 8$). When $h_0 = 3$, it is cheaper to hold inventory at the warehouse and the retailers' ignorance can be compensated by keeping more inventory at the warehouse. However, when $h_0 = 8$, it is expensive to hold inventory at the warehouse. Hence, the system incurs very high backordering costs by not having sufficient inventories at the warehouse or at the retailers.

When $h_0 = 3$, it costs more if retailers ignore the disruptions in their supply processes compared to the disruptions in the supply process of the warehouse (3.80% vs. 10.49%). When retailers ignore the disruptions in the supply process of the warehouse, the warehouse keeps more inventory. However, it does not make sense for the warehouse to keep more inventory as a precaution for the disruptions in the supply processes of the retailers, because it cannot send these inventories to the retailers during the disruptions. This is why the retailers' ignorance of their own disruptions is more costly. On the other hand, when $h_0 = 8$, it costs more if retailers ignore the disruptions in the supply system of the warehouse. The reason is that the warehouse does not want to keep more inventory to protect against these disruptions. It is costly to do so. The better option is to incur higher backorder costs and, overall, this implies a higher total cost for the whole system.

When all the locations ignore the disruptions at the warehouse, the optimal base-stock level for the warehouse is 0 for both h_0 and h_r. In addition, the optimal retailer base-stock levels are the same for both cases. At optimality, s_0 can be positive for the case $h_0 = 3$ but it is always 0 when $h_0 = 8$. For $h_0 = 3$, having less inventory at the warehouse results in higher backorders. This is why, when all the locations ignore the disruptions at the warehouse, the percentage cost increase is higher when $h_0 = 3$ (20.31% vs. 15.14%).

When all the locations ignore the disruptions at the retailers, the percentage cost increase is higher for the case with $h_0 = 3$ (24.67% vs. 9.51%). The locations only keep more inventory to protect gainst the warehouse disruptions. The numerical results suggest that the extra inventory is kept only at the warehouse when $h_0 = 3$ and it is kept at the retailers when $h_0 = 8$. Keeping inventories at a location away from customers implies higher backorder costs, since during disruptions at the retailers, these inventories cannot be sent to the retailers.

Overall, based on these results and discussions, we can provide the following guidelines for company managers:

- Locations closer to the customers should never ignore the disruptions.
- If holding inventory at the upstream locations is expensive, the disruptions in the supply processes of the upstream locations should not be ignored by the downstream locations.

- If holding inventory at the upstream locations is cheap, the percentage cost increase of ignoring the upstream disruptions at the downstream locations is not that large. Hence, if downstream locations choose to ignore some of the disruptions, these can be the disruptions in the supply processes of the upstream locations but not the disruptions in their own supply systems.
- Ignoring all the disruptions in all the supply systems can be very costly. The system can benefit if downstream locations take into consideration at least some of the disruptions.

5.3.2. Disruption and Recovery Probabilities Given the results in Section 5.3.1, suppose that a company decides not to ignore the disruptions. Then, in addition to adjusting its base-stock levels, the company can work on either reducing the disruption probabilities or increasing the recovery probabilities. In order to see the effects of these changes, we analyze a serial system with one warehouse and one retailer.[5] We use exactly the same data as in Section 5.3.1, with $h_0 = 3$. As a base case, we assume that all probabilities are equal to 0.5. We consider percentage improvements in all four probabilities and generate Figure 5.

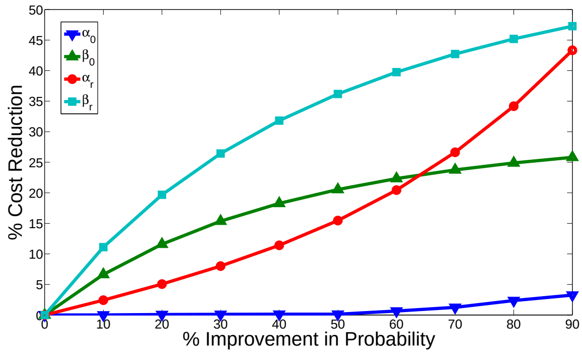


Figure 5 **Percentage cost reduction achieved by reducing the disruption and increasing the recovery probabilities**

Figure 5 suggests that by increasing the recovery probabilities, especially of the disruptions in the supply processes of the retailers, substantial cost benefits can be achieved. Decreasing the disruption probability in the supply processes of the retailers can lead to cost improvements (at most 30% if the disruption probability is reduced from 0.9 to 0.1) as well. However, trying to reduce the disruption probabilities in the supply process of the warehouse does not provide any benefit.

As a result, we advise company managers to focus on reducing the duration of all the disruptions and reducing the probability of occurrence of disruptions closer to the customers. Instead of trying to avoid the disruptions, companies need to develop strategies to mitigate their effects as quickly as possible. The most benefit can be achieved from concentrating on the disruptions happening in the supply processes of the locations closer to customers.

6. Conclusions and Future Research Directions

In this paper, we analyze a deterministic-demand OWMR system subject to random disruptions. We assume that the only type of randomness involved in the system is supply disruptions. Among the many ways to mitigate the effects of the disruptions, we consider inventory mitigation.

Initially, we assume that all the retailers are identical. We study three different scenarios, with disruptions happening only in the supply process of the warehouse, only in the supply processes of the retailers and in both of the supply processes. For the first case, we show that the base-stock

[5] The conclusions in this section are independent of the number of retailers in the system. This is why we choose to consider the simplest OWMR system, i.e., a two-location serial system.

levels that minimize the total expected cost depend on the relative magnitudes of the unit holding costs. If it is cheap to hold inventory at the warehouse, then this is the location to protect against the disruptions in its supply system. The opposite is true when the unit holding cost of the retailers is smaller than the warehouse unit holding cost. When disruptions only happen in the supply processes of the retailers, the warehouse sends the orders but the retailers cannot receive them. Since the warehouse is charged the holding cost of these items, it is never optimal for it to hold more than one period's worth of demand. For this case, the retailers hold more inventory to protect against disruptions. When disruptions happen in the supply processes of both the warehouse and the retailers, both locations tend to keep more inventory. We propose a heuristic procedure to approximate the base-stock levels of all locations. Although we have not yet been able to prove this approach is exact, the heuristic finds the optimal solution in every instance tested.

We then relax our assumption of identical retailers and study the three scenarios mentioned above. We propose a heuristic procedure to obtain the base-stock levels of all the locations, when disruptions happen only in the supply system of the warehouse. The average percentage difference between the optimal cost and the cost suggested by the heuristic is 0.60%, and in 87.96% of the cases the heuristic gives the optimal results. We obtain exact expressions for the optimal base-stock levels when disruptions only happen in the supply processes of the retailers. These expressions suggest that each retailer holds more inventory to protect against disruptions in its own supply process. Finally, we propose a heuristic to solve the problem with disruptions at both locations. The heuristic relies on the idea of decomposing the system into serial systems, solving for the base-stock levels (using the results for identical retailers' case) and aggregating the system back to the original OWMR system by summing the base-stock levels of all parts of the warehouse in the serial systems. The average percentage difference between the optimal cost and the cost suggested by the heuristic is 0.58%, and in 87.26% of the cases the heuristic gives the optimal results.

Via numerical analysis, we study the sensitivity of the optimal base-stock levels and the corresponding expected costs to the changes in disruption and recovery parameters. In addition, we quantify the costs of ignoring the disruptions at different parts of the supply chain. We conclude that if companies choose to ignore and do nothing to prevent the consequences of some of the disruptions, these should not be the ones happening close to the customers. In addition, we conclude that it is more costly for the system if supply chain players that have direct contact with the customers ignore the disruptions. By analyzing the effects of decreasing the disruption probabilities and increasing the recovery probabilities, we advise companies to work on reducing the duration of the supply disruptions instead of trying to prevent them.

In this paper we made a few assumptions in particular, deterministic demand and no leadtimes between the warehouse and the retailers. We believe that these simplifying assumptions enabled us to obtain results and insights which will assist us in solving more complex problems. In fact, we have already relaxed our assumption of non-overlapping disruptions and modeled the system as a DTMC. Solving this model and analyzing the effects of overlapping disruptions is one possibility for future research. Our preliminary results with this model suggest that when we add another form of uncertainty to the system subject to random disruptions, the expressions for the expected cost functions and the optimal base-stock levels are even harder to obtain. Therefore, relaxing the assumption of deterministic demand might be accompanied with a need for other approximations and heuristics. In addition, the models can be extended by adding more echelons and considering different disruption processes.

Acknowledgments

Appendix. Proofs

A. Proof of Theorem 3.1

As s_0 changes, $C(s_0,s_r)$ changes linearly until s_0 equals a multiple of Nd. Similarly, as s_r changes, $C(S_0,s_r)$ changes linearly until s_r equals a multiple of d. The latter result is obvious for the first and third components of $C(s_0,s_r)$. For the second component, it relies on s_0 being a multiple of Nd, hence $\frac{1}{N}(iNd-s_0)^+$ being a multiple of d.

B. Proof of Theorem 3.1

For convexity, we need to show that the second-order difference with respect to s_r, i.e., $\Delta^2_{s_r}C(s_0,s_r)$, is non-negative, i.e., $\Delta^2_{s_r}C(s_0,s_r)\geq 0$. Since s_r^* is an integer multiple of d, we define $\Delta^2_{s_r}C(s_0,s_r)$ as

$$\Delta^2_{s_r}C(s_0,s_r)=\Delta_{s_r}C(s_0,s_r+d)-\Delta_{s_r}C(s_0,s_r)$$

where

$$\Delta_{s_r}C(s_0,s_r)=C(s_0,s_r+d)-C(s_0,s_r).$$

$$\begin{aligned}
\Delta_{s_r}C(kNd,s_r)=&\pi_{0,0}\left[h_0kNd+N\left(h_r\left[s_r+d-d\right]^+ +p_r\left[d-s_r-d\right]^+\right)\right]\\
&-\pi_{0,0}\left[h_0kNd+N\left(h_r\left[s_r-d\right]^+ +p_r\left[d-s_r\right]^+\right)\right]\\
&+\sum_{i=1}^{\infty}\pi_{i,0}\left[h_0\left[kNd-iNd\right]^+ +N\left(h_r\left[s_r+d-\frac{1}{N}(iNd-kNd)^+-d\right]^+ +p_r\left[d+\frac{1}{N}(iNd-kNd)^+-s_r-d\right]^+\right)\right]\\
&-\sum_{i=1}^{\infty}\pi_{i,0}\left[h_0\left[kNd-iNd\right]^+ +N\left(h_r\left[s_r-\frac{1}{N}(iNd-kNd)^+-d\right]^+ +p_r\left[d+\frac{1}{N}(iNd-kNd)^+-s_r\right]^+\right)\right]\\
&+\sum_{j=1}^{\infty}\pi_{0,j}\left[h_0\left(kNd+jNd\right)+N\left(h_r\left[s_r+d-(j+1)d\right]^+ +p_r\left[(j+1)d-s_r-d\right]^+\right)\right]\\
&-\sum_{j=1}^{\infty}\pi_{0,j}\left[h_0\left(kNd+jNd\right)+N\left(h_r\left[s_r-(j+1)d\right]^+ +p_r\left[(j+1)d-s_r\right]^+\right)\right]\\
=&N\left(\pi_{0,0}+\sum_{i=1}^{k}\pi_{i,0}\right)\left[h_r\left[s_r\right]^+ -h_r\left[s_r-d\right]^+ +p_r\left[-s_r\right]^+ -p_r\left[d-s_r\right]^+\right]\\
&+Nh_r\sum_{i=k+1}^{\infty}\pi_{i,0}\left[\left[s_r-\frac{1}{N}(iNd-kNd)\right]^+ -\left[s_r-\frac{1}{N}(iNd-kNd)-d\right]^+\right]\\
&+Np_r\sum_{i=k+1}^{\infty}\pi_{i,0}\left[\left[\frac{1}{N}(iNd-kNd)-s_r\right]^+ -\left[d+\frac{1}{N}(iNd-kNd)-s_r\right]^+\right]\\
&+Ndh_r\sum_{j=1}^{\frac{s_r}{d}-1}\pi_{0,j}-Ndp_r\sum_{j=\frac{s_r}{d}}^{\infty}\pi_{0,j}\\
=&N\left(\pi_{0,0}+\sum_{i=1}^{k}\pi_{i,0}\right)\left[h_r\left[s_r\right]^+ -h_r\left[s_r-d\right]^+ +p_r\left[-s_r\right]^+ -p_r\left[d-s_r\right]^+\right]\\
&+Ndh_r\sum_{i=k+1}^{\frac{s_r}{d}+(k-1)}\pi_{i,0}-Ndp_r\sum_{i=\frac{s_r}{d}+k}^{\infty}\pi_{i,0}+Ndh_r\sum_{j=1}^{\frac{s_r}{d}-1}\pi_{0,j}-Ndp_r\sum_{j=\frac{s_r}{d}}^{\infty}\pi_{0,j}
\end{aligned}$$

Given this, we can write $\Delta^2C(s_0,s_r)$ as follows:

$$\begin{aligned}
\Delta^2_{s_r}C(kNd,s_r)=&N\left(\pi_{0,0}+\sum_{i=1}^{k}\pi_{i,0}\right)\left[h_r\left(d-s_r\right)+h_r\left[s_r-d\right]^+ +p_r\left[d-s_r\right]^+\right]\\
&+Nd\left(h_r+p_r\right)\left[\pi_{\frac{s_r}{d}+k,0}+\pi_{0,\frac{s_r}{d}}\right]
\end{aligned}$$

If $s_r=0$,

$$\Delta^2_{s_r}C(kNd,s_r)=Nd(h_r+p_r)\left(\pi_{0,0}+\sum_{i=1}^{k}\pi_{i,0}\right)+Nd\left(h_r+p_r\right)\left[\pi_{k,0}+\pi_{0,0}\right]$$

≥ 0

If $s_r \geq d$,

$$\Delta^2_{s_r} C(kNd, s_r) = Nd\Big(h_r + p_r\Big)\Big[\pi_{\frac{s_r}{d}+k,0} + \pi_{0,\frac{s_r}{d}}\Big] \geq 0.$$

Therefore, given that the warehouse base-stock level is kNd, the cost function $C(kNd, s_r)$ is convex in s_r and $s^*_r(kNd)$ is the smallest s_r to satisfy the inequality $\Delta C(kNd, s_r) \geq 0$. If $s_r = 0$, we have $s^*_r(kNd) \neq 0$, since

$$\Delta_{s_r} C(kNd, s_r) = N\Big(\pi_{0,0} + \sum_{i=1}^{k} \pi_{i,0}\Big)\Big[-p_r\Big] - Ndp_r \sum_{i=k}^{\infty} \pi_{i,0} - Ndp_r \sum_{j=0}^{\infty} \pi_{0,j} < 0.$$

Hence, $s^*_r(kNd) \neq 0$. Given that $s^*_r(kNd) \geq d$, $\Delta_{s_r} C(kNd, s_r)$ reduces to

$$\begin{aligned}\Delta_{s_r} C(kNd, s_r) &= Ndh_r\Big(\pi_{0,0} + \sum_{i=1}^{\frac{s_r}{d}+(k-1)} \pi_{i,0} + \sum_{j=1}^{\frac{s_r}{d}-1} \pi_{0,j}\Big) - Ndp_r\Big(\sum_{i=\frac{s_r}{d}+k}^{\infty} \pi_{i,0} + \sum_{j=\frac{s_r}{d}}^{\infty} \pi_{0,j}\Big)\\ &= Ndh_r\Big(\pi_{0,0} + \sum_{i=1}^{\frac{s_r}{d}+(k-1)} \pi_{i,0} + \sum_{j=1}^{\frac{s_r}{d}-1} \pi_{0,j}\Big) - Ndp_r\Big(1 - \pi_{0,0} - \sum_{i=1}^{\frac{s_r}{d}+(k-1)} \pi_{i,0} - \sum_{j=1}^{\frac{s_r}{d}-1} \pi_{0,j}\Big).\end{aligned}$$

Hence, the optimal retailer base-stock level is $s^*_r(kNd) = m^*d$, where $m^* \geq 1$ is the smallest integer m to satisfy the inequality

$$\pi_{0,0} + \sum_{i=1}^{m+(k-1)} \pi_{i,0} + \sum_{j=1}^{m-1} \pi_{0,j} \geq \frac{p_r}{p_r + h_r}.$$

C. Proof of Lemma 3.1

If disruptions only happen at the supply process of the warehouse, we have $\pi_{i,j} = 0\ \forall j > 0$. Hence, (2) reduces to $\pi_{0,0} + \sum_{i=0}^{m^*+k-1} \pi_{i,0} \geq \frac{p_r}{p_r+h_r}$. By definition, the left-hand side of the inequality is $F(m^* + k - 1)$.

Similarly, if disruptions only happen at the supply processes of the retailers, we have $\pi_{i,j} = 0\ \forall i > 0$. Hence, (2) reduces to $\pi_{0,0} + \sum_{i=0}^{m^*-1} \pi_{i,0} \geq \frac{p_r}{p_r+h_r}$. By definition, the left-hand side of the inequality is $G(m^* - 1)$.

D. Proof of Proposition 3.1

$$\begin{aligned}\Delta_{s_0} C(s_0, s_r) &= \big(\pi_{0,0} + \sum_{j=1}^{\infty} \pi_{0,j}\big)h_0(s_0 + Nd) - \big(\pi_{0,0} + \sum_{j=1}^{\infty} \pi_{0,j}\big)h_0 s_0\\ &\quad + \sum_{i=1}^{\infty} \pi_{i,0}\Big[h_0\Big[s_0 + Nd - iNd\Big]^+ + N\Big(h_r\Big[s_r - \frac{1}{N}(iNd - s_0 - Nd)^+ - d\Big]^+ + p_r\Big[d + \frac{1}{N}(iNd - s_0 - Nd)^+ - s_r\Big]^+\Big)\Big]\\ &\quad - \sum_{i=1}^{\infty} \pi_{i,0}\Big[h_0\Big[s_0 - iNd\Big]^+ + N\Big(h_r\Big[s_r - \frac{1}{N}(iNd - s_0)^+ - d\Big]^+ + p_r\Big[d + \frac{1}{N}(iNd - s_0)^+ - s_r\Big]^+\Big)\Big]\\ &= Nd\Big(h_0\Big(\pi_{0,0} + \sum_{j=1}^{\infty} \pi_{0,j} + \sum_{i=1}^{\frac{s_0}{Nd}} \pi_{i,0}\Big) + h_r \sum_{i=\frac{s_0}{Nd}+1}^{\frac{s_0}{Nd}+\frac{s_r}{d}-1} \pi_{i,0} - p_r \sum_{i=\frac{s_0}{Nd}+\frac{s_r}{d}}^{\infty} \pi_{i,0}\Big)\\ \Delta^2_{s_0} C(s_0, s_r) &= Nd\Big(h_0\Big(\pi_{0,0} + \sum_{j=1}^{\infty} \pi_{0,j} + \sum_{i=1}^{\frac{s_0+Nd}{Nd}} \pi_{i,0}\Big) + h_r \sum_{i=\frac{s_0+Nd}{Nd}+1}^{\frac{s_0+Nd}{Nd}+\frac{s_r}{d}-1} \pi_{i,0} - p_r \sum_{i=\frac{s_0+Nd}{Nd}+\frac{s_r}{d}}^{\infty} \pi_{i,0}\Big)\\ &\quad - Nd\Big(h_0\Big(\pi_{0,0} + \sum_{j=1}^{\infty} \pi_{0,j} + \sum_{i=1}^{\frac{s_0}{Nd}} \pi_{i,0}\Big) + h_r \sum_{i=\frac{s_0}{Nd}+1}^{\frac{s_0}{Nd}+\frac{s_r}{d}-1} \pi_{i,0} - p_r \sum_{i=\frac{s_0}{Nd}+\frac{s_r}{d}}^{\infty} \pi_{i,0}\Big)\\ &= Nd\Big(\Big(h_0 - h_r\Big)\pi_{\frac{s_0}{Nd}+1,0} + \Big(h_r + p_r\Big)\pi_{\frac{s_0}{Nd}+\frac{s_r}{d},0}\Big)\end{aligned}$$

E. Proof of Theorem 3.1

For $h_r \leq h_0$, $\Delta^2_{s_0} C(s_0, s_r) \geq 0$, implying the convexity of $C(s_0, s_r)$ in s_0.

F. Proof of Theorem 3.2

Define t^* as

$$t^* = \min\left\{t : t \in \mathbb{Z}, t \geq F^{-1}\left(\frac{p_r}{p_r + h_r}\right)\right\}$$

Given this, $s_r^*(kNd)$ is $\max\{d, (t^* - k + 1)d\}$. The cost function, $C(s_0, s_r)$ on page 7, can be written as follows ($\pi_{i,j} = 0\ \forall j \geq 1$):

$$\begin{aligned} C(kNd, s_r^*(kNd)) &= \sum_{i=0}^{\infty} \pi_i \left[h_0 \left[s_0 - iNd\right]^+ + N\left(h_r \left[s_r^*(kNd) - \frac{1}{N}(iNd - s_0)^+ - d\right]^+ + p_r \left[d + \frac{1}{N}(iNd - s_0)^+ - s_r^*(kNd)\right]^+ \right)\right] \\ &= Nd \sum_{i=0}^{\infty} \pi_i \left[h_0 \left[k - i\right]^+ + \left(h_r \left[\max\{1, t^* - k + 1\} - (i-k)^+ - 1\right]^+ + p_r \left[1 + (i-k)^+ - \max\{1, t^* - k + 1\}\right]^+ \right)\right] \end{aligned}$$

For $k \geq t^*$, the functions $C(kNd, s_r^*(kNd)), \Delta_{s_0} C(kNd, s_r^*(kNd))$ and $\Delta_{s_0}^2 C(kNd, s_r^*(kNd))$ are as follows:

$$\begin{aligned} C(kNd, s_r^*(kNd)) &= Nd\left[h_0 \sum_{i=0}^{k} \pi_i (k-i) + p_r \sum_{i=k+1}^{\infty} \pi_i (i-k) \right] \\ \Delta_{s_0} C(kNd, s_r^*(kNd)) &= Nd(h_0 + p_r) F(k) - Ndp_r \\ \Delta_{s_0}^2 C(kNd, s_r^*(kNd)) &= Nd(h_0 + p_r)\pi_{k+1} \geq 0 \end{aligned}$$

Hence, when $k \geq t^*$, $C(kNd, s^*(kNd))$ is convex in $s_0 = kNd$. For $k \leq t^* - 1$, the functions $C(kNd, s_r^*(kNd)), \Delta_{s_0} C(kNd, s_r^*(kNd))$ and $\Delta_{s_0}^2 C(kNd, s_r^*(kNd))$ are as follows:

$$\begin{aligned} C(kNd, s_r^*(kNd)) &= Nd\left[h_0 kF(k) - h_0 \sum_{i=0}^{k} i\pi_i + h_r (t^* - k) F(k) + h_r \sum_{i=k+1}^{\infty} \pi_i (t^* - i)^+ + p_r \sum_{i=k+1}^{\infty} \pi_i (i - t^*)^+ \right] \\ &= Nd\left[h_0 kF(k) - h_0 \sum_{i=0}^{k} i\pi_i + h_r t^* F(k) - h_r kF(k) + h_r \sum_{i=k+1}^{t^*} \pi_i (t^* - i) + p_r \sum_{i=t^*}^{\infty} \pi_i (i - t^*) \right] \\ &= Nd\left[(h_0 - h_r) kF(k) - h_0 \sum_{i=0}^{k} i\pi_i + (h_r + p_r) t^* F(t^*) - p_r t^* - h_r \sum_{i=k+1}^{t^*} i\pi_i + p_r \sum_{i=t^*}^{\infty} i\pi_i \right] \end{aligned}$$

$$\begin{aligned} \Delta_{s_0} C(kNd, s_r^*(kNd)) &= Nd(h_0 - h_r) F(k) \\ \Delta_{s_0}^2 C(kNd, s_r^*(kNd)) &= Nd(h_0 - h_r)\pi_{k+1} \end{aligned}$$

Hence, when $k \leq t^* - 1$, $C(kNd, s_r^*(kNd))$ is non-increasing and concave in s_0 if $h_0 \leq h_r$ and it is non-decreasing and convex in s_0 if $h_0 \geq h_r$.

G. Proof of Theorem 3.2

When $h_0 \leq h_r$, the cost function $C(s_0, s_r^*(s_0))$ is decreasing up to $t^* - 1$ and convex afterward. Therefore, the minimizer of this function is the minimizer of the convex part of the function. Hence, we have $s_0 = k^* Nd$, where k^* is the smallest integer k that satisfies the inequality $\Delta C(knd, s_r^*(kNd)) = Nd(h_0 + p_r)F(k-1) - Ndp_r \geq 0$.

When $h_0 \geq h_r$, the cost function $C(s_0, s_r^*(s_0))$ is convex in both regions. We know that the minimum warehouse base-stock level is Nd. Note that $k = 1$ is the smallest integer to satisfy $Nd(h_0 - h_r)F(k-1) - h_r \pi_{k+1} \geq 0$. Hence, $s_0^* = Nd$.

H. Proof of Theorem 3.4.1

The result follows directly from Proposition 3.1 and Theorem 3.1.

I. Proof of Theorem 3.4.1

Based on Proposition 3.1, where $\Delta_{s_0} C(s_0, s_r)$ is defined, and Section 8, where $\Delta_{s_r} C(s_0, s_r)$ is determined, we can determine $\Delta_{s_0, s_r}^2 C(s_0, s_r)$ and $\Delta_{s_r, s_0}^2 C(s_0, s_r)$ as follows:

$$\Delta_{s_0, s_r}^2 C(s_0, s_r) = \Delta_{s_r, s_0}^2 C(s_0, s_r) = Nd(h_r + p_r)\pi_{\frac{s_0}{Nd} + \frac{s_r}{d}, 0}$$

We have $H = \begin{vmatrix} \Delta^2_{s_0} C(s_0,s_r) & \Delta^2_{s_0,s_r} C(s_0,s_r) \\ \Delta^2_{s_r,s_0} C(s_0,s_r) & \Delta^2_{s_r} C(s_0,s_r) \end{vmatrix}$

$$\begin{aligned}
\Delta^2_{s_0} C(s_0,s_r) &= Nd\Big(h_0 - h_r\Big)\pi_{\frac{s_0}{Nd}+1,0} + Nd\Big(h_r + p_r\Big)\pi_{\frac{s_0}{Nd}+\frac{s_r}{d},0} \\
\Delta^2_{s_0,s_r} C(s_0,s_r) &= \Delta^2_{s_r,s_0} C(s_0,s_r) = Nd\Big(h_r + p_r\Big)\pi_{\frac{s_0}{Nd}+\frac{s_r}{d},0} \\
\Delta^2_{s_r} C(s_0,s_r) &= Nd\Big(h_r + p_r\Big)(\pi_{\frac{s_0}{Nd}+\frac{s_r}{d},0} + \pi_{0,\frac{s_r}{d}})
\end{aligned}$$

$\Delta^2_{s_0} C(s_0,s_r)$ and $\Delta^2_{s_r} C(s_0,s_r)$ come from Proposition 3.1 and Section 8, respectively. Since $\Delta^2_{s_0} C(s_0,s_r) \geq 0$ and $\Delta^2_{s_0} C(s_0,s_r)\Delta^2_{s_r} C(s_0,s_r) \geq \Big(\Delta^2_{s_0,s_r} C(s_0,s_r)\Big)^2$, H is positive semi-definite.

J. Proof of Lemma 5

The DTMC representing the disruptions in the supply processes of both the warehouse and the retailers has the following transition probabilities: $P((0,0),(1,0)) = \alpha_0$, $P((0,0),(0,1)) = \alpha_r$, $P((0,0),(0,0)) = 1-\alpha_0-\alpha_r$, $P((i,0),(i+1,0)) = 1-\beta_0\ \forall i \geq 1$, $P((i,0),(0,0)) = \beta_0\ \forall i \geq 1$, $P((0,j),(0,j+1)) = 1-\beta_r\ \forall j \geq 1$ and $P((0,j),(0,0)) = \beta_r\ \forall j \geq 1$. The set of equations we need to solve to obtain the limiting probabilities are as follows:

$$\begin{aligned}
\pi_{0,0} &= (1-\alpha_0-\alpha_r)\pi_{0,0} + \beta_0 \sum_{i=1}^{\infty} \pi_{i,0} + \beta_r \sum_{j=1}^{\infty} \pi_{0,j} \\
\pi_{1,0} &= \alpha_0 \pi_{0,0} \\
\pi_{i,0} &= (1-\beta_0)\pi_{i-1,0} \qquad \forall i \geq 2 \\
\pi_{0,1} &= \alpha_r \pi_{0,0} \\
\pi_{0,j} &= (1-\beta_r)\pi_{0,j-1} \qquad \forall j \geq 2
\end{aligned}$$

In addition, we have $\pi_{0,0} + \sum_{i=1}^{\infty} \pi_{i,0} + \sum_{j=1}^{\infty} \pi_{0,j} = 1$. Via simple algebra, the steady state probabilities are obtained.

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