Infinite-Horizon Models for Inventory Control under Yield Uncertainty and Disruptions*

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Abstract

We consider a firm facing supply chain risk in two forms: disruptions and yield uncertainty. We demonstrate the importance of analyzing a sufficiently long time horizon when modeling inventory systems subject to supply disruptions. Several previous papers have used single-period newsboy-style models to study supply disruptions, and we show that such models underestimate the risk of supply disruptions and generate sub-optimal solutions. We consider one case where a firm's only sourcing option is an unreliable supplier subject to disruptions and yield uncertainty, and a second case where a second, reliable (but more expensive) supplier is available. We develop models for both cases to determine the optimal order and reserve quantities. We then compare these results to those found when a single-period approximation is used. We demonstrate that a single-period approximation causes increases in cost, under-utilizes the unreliable supplier, and distorts the order quantities that should be placed with the reliable supplier in the two-supplier case. Moreover, using a single-period model can lead to selecting the wrong strategy for mitigating supply risk.

Key Words: supply chain risk management, supply disruptions, yield uncertainty, dual-sourcing, inventory management

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1 Introduction

The need for quantitative models addressing supply chain risk management is growing with the global expansion of supply chains. In this paper, we provide such a quantitative model by focusing on supply uncertainty, a particularly important aspect of supply chain risk and one that has received a great deal of attention in both the research and practitioner communities in recent years. Literature on inventory management contains numerous single-period models, and these models often provide excellent results. However, when supply disruptions are possible, single-period models can grossly underestimate the risk that disruptions pose to the system. In this paper we demonstrate that single-period models do not generate solutions that provide enough protection from disruptions.

We consider two types of supply uncertainty: yield uncertainty and disruptions. Yield uncertainty occurs when the quantity of supply delivered is a random variable, typically modeled as either a random additive or multiplicative quantity. Disruptions occur when supply is subject to partial or complete failure. Disruptions can be more difficult to analyze than yield uncertainty because the state variables (e.g., inventory level) are typically more strongly correlated over time under disruptions than under yield uncertainty. Truncating the time horizon to consider only a single period makes this analysis easier and may allow for closed-form results; several papers employ this approach [e.g., 11, 12, 39]. However, as we demonstrate, truncating also underestimates the disruption risk.

Analysis of systems with yield uncertainty provides solutions that parallel classical newsboy results; that is, the optimal order level is the sum of the mean demand (cycle stock) plus an additional amount determined by the cost and variability parameters (safety stock). However, managers have other options besides using safety stock to mitigate supply uncertainty. Other strategies include acceptance, when protecting against supply uncertainty is too costly and the best policy is to ignore it, and mitigation through the use of backup suppliers [37], product substitution [5], or other alternatives to satisfy demand.

Supply uncertainty has gained increased attention in recent years. Notable events such as the 9/11 terrorist attacks or major natural disasters have brought focus to supply chain disruption studies. One example of how a company can recover quickly from disruptions when proper disruption management techniques are used is Wal-Mart's performance after the Hurricane Katrina disaster in the the Gulf coast. Wal-Mart has personnel dedicated to tracking potential disruptions and planning for or coping with them. With Katrina approaching, Wal-Mart overstocked its nearby distribution centers with items it knew would be needed (such as bottled water, Pop-Tarts, and generators), and after Katrina struck, its strong transportation network allowed it to respond quickly to deliver supplies and reduce the disruption to its supply chain. Without this planning, Wal-Mart's recovery time would have been much longer and much more costly for the company [18]. Home Depot, too, had learned from past hurricanes and was better prepared to handle the demand after the disruption caused by Katrina. They stocked up on supplies that would be demanded by customers, as well as supplies necessary to get their stores back up and running [7]. Both of these companies continually address disruption risk and had policies in place for coping with the impending disaster. This proactive planning for disruptions allowed them to efficiently prepare for and recover from the hurricane.

Supply disruptions can also be caused by factors other than major catastrophes. More common incidents such as snow storms, customs delays, fires, strikes, slow shipments, etc. can halt production and/or transportation capability, causing lead time delays that disrupt material flow. As supply chains grow globally, there are higher chances for such delays. Capacity shortfalls at a supplier may also cause disruptions, particularly if a firm is not the supplier's highest-priority customer. Sometimes disruptions are a planned part of a supplier-retailer relationship based on contracted material availability. If, for example, a supplier promises an 80% material availability in their contract with a retailer, then the retailer can anticipate that its supply will be unavailable up to 20% of the time. Additionally, suppliers may be internal to a firm, and improving process reliability may be more costly than mitigating disrupted supply through inventory use.

Ultimately, supply disruptions are not uncommon, and firms must anticipate them. Not all firms have the buying power that Wal-Mart and Home Depot do to plan for impending disruptions, and not all disruptions have advanced warning. Thus ongoing mitigation policies must be considered, and we model disruptions with a stationary probability of occurrence in this paper. This could represent a single disruption source or multiple aggregated sources, but it characterizes the ongoing risk that a firm must continually anticipate.

One paper that models supply uncertainty in a single-period setting is that of Dada et al. [12]. The authors model a retailer with multiple unreliable suppliers in a newsboy setting, where inventory (from one or multiple suppliers) is used to mitigate generally distributed supply uncertainty, and they include disruptions as a possible realization. While their model yields excellent results for continuous supply distributions, we will demonstrate that if the firm is capable of planning proactively for future periods and the disruption risk is significant (the penalty costs for shortages are high and/or disruptions have a high probability of occurrence), the optimal base-stock levels are underestimated by single-period models. Tomlin and Wang [39] also employ a single-period model to examine dual-sourcing and mix-flexibility decisions when disruptions are present, as do Babich et al. [4] for multiple competing suppliers. We will demonstrate that single-period models also underestimate the need for back up suppliers, because they do not adequately capture the long-term risk of stocking out for multiple periods.

Another single-period model for supply disruptions is presented by Chopra, Reinhardt, and Mohan [11]. The authors consider yield uncertainty as well as disruptions in a single-period setting. They compare their optimal solution to that of a "bundled" solution, in which disruptions are not separately accounted for and the aggregate variance of the delivered quantity (accounting for both yield uncertainty and disruptions) is used for the solution. They demonstrate the error and increased costs incurred by bundling the two sources of uncertainty. We stress that one should not, as some authors have, interpret Chopra et al.'s work as a justification for ignoring one source of uncertainty when both are present (i.e., for optimizing them separately). Rather, the paper highlights the necessity of accounted for the two types of uncertainty correctly, as multiple factors affecting the same system, rather than treating them as a single random process.

In this paper, we illustrate the importance of considering multiple time periods when disruptions are present, using a model similar to that of Chopra et al. [11]. Our insights also apply, at least approximately, to other single-period disruption models, such as those of Dada et al. [12] or Tomlin and Wang [39].

Disruptions have a significant impact on future periods, and planning for these disruptions can have a significant impact on order quantities. Therefore we consider a multiple-period (in particular, infinite-horizon) model. Note that single-period models may be appropriate in some cases; for example, if product is perishable and can only be used in one period, or if a period is very long (e.g., the length of an entire selling season) and only one order may be placed to cover the season. However, single-period models are also used to find approximate solutions for the case when inventory may be carried period-to-period, and disruptions may last for relatively short periods. Therefore we focus on a system where orders may be placed periodically and disruptions may occur in any period, and assume that the length of one period is the same in either the single- or multi-period model. This allows us to demonstrate the importance of considering multiple periods if inventory may be carried and proactive measures may be taken.

We consider both a system with one unreliable supplier, as well as a system with one unreliable supplier and a second, perfectly reliable (but more expensive) supplier. We compare the costs of the system if the exact cost is used to optimize the decision variables with the cost if a single-period model is used. We demonstrate that our infinite-horizon model leads to solutions that provide lower expected costs, and that single-period models can lead to incorrect overall strategies for supply risk mitigation (e.g., acceptance instead of mitigation using alternate suppliers).

The remainder of the paper is structured as follows: Section 2 reviews the related literature, Section 3 presents the analysis for a single supplier and numerical comparisons of the exact and approximate solutions, Section 4 presents the same for the case in which a second supplier is available, and Section 5 presents conclusions and insights.

2 Literature

In this section we highlight literature focused on supply chain risk management, specifically focusing on papers considering mitigating the risk of supply disruptions or yield uncertainty. The terrorist attacks in 2001 motivated literature focusing on catastrophic disruption risk. Snyder et al. [32] thoroughly review supply disruption literature and discuss the significant growth of the field over the past decade. Sheffi [28, 29] and Simchi-Levi et al. [30] stress the importance of sharing risk throughout the supply chain and the dangers of disruptions to just-in-time (JIT) systems. They indicate that JIT systems can lack buffers for supply uncertainty and can be at high risk for interruption of material flow. Lean methods in supply chain management advocate the reduction of excess suppliers, but if a company reduces to too narrow of a supplier base it leaves that company at risk if something happens to disrupt the production of one or all of those few suppliers.

Tang [36] provides a review of initiatives in supply chain risk management and cites the need for more quantitative models. He suggests that global growth of supply chains has made them more susceptible to uncertainties, and that more models are needed to bridge the gap between operations research theory and the practice of managing supply uncertainty risk. Chapman et al. [8] also cite the need for operations models to capture more supply chain vulnerabilities. They describe supply chain vulnerability as a complex puzzle that must be pieced together to view the big picture in order to identify how to mitigate risks for the system as a whole. Typically the literature on supply uncertainty assumes that supply is either subject to complete disruptions or yield uncertainty. Yield uncertainty may be dependent on or independent of the actual order quantity. Gerchak et al. [14] discuss multiplicative yield, where the order quantity is multiplied by a random variable, for one supplier over one and two periods. Dada et al. [12] extend the classical one-period, stochastic-demand newsboy model to consider multiple suppliers who may produce random supply based on the quantities ordered, and establish rules for when it is worthwhile to use less-reliable suppliers if they have low enough unit costs. Federgruen and Yang [13] also consider multiple suppliers subject to yield uncertainty in a one-period model and develop an approximation for the expected shortfall. Yano and Lee [40] present a thorough review of yield uncertainty models. They identify the three challenges of modeling these systems as modeling the costs, the yield itself, and the performance characteristics. The models they review include general models, single-stage continuous-time models, discrete-time models, and complex manufacturing systems (under which multi-period models are classified). They stress that these systems are very complex and since closed-form solutions are often unachievable, valid heuristics must be further studied and developed.

Supply disruptions often complicate inventory models significantly. Typically disruptions are modeled as events which occur randomly and may have a random length; in this way the supplier can be modeled as either *on* or *off*. Parlar and Berkin [22] introduce disruptions in the EOQ model (corrections for their model were published by Berk and Arreola-Risa [6]). Parlar and Perry [23, 24] extend the uncertain-supply EOQ model to allow for non-zero reorder points, and introduce multiple suppliers to the model. Snyder [31] introduces an approximation for Parlar and Berkin's uncertain-supply EOQ model which, unlike the exact model, can be solved in closed form.

Non-EOQ models with disruptions have also been examined. Schmitt et al. [27] prove several results for a periodic-review base-stock inventory system that we will draw on in this paper. Gupta [15] examines a (Q, r) system with Poisson-distributed demand, a supplier subject to disruptions, and lost sales. He discusses the impact of lead times and indicates that they can be the dominant factor in setting optimal parameter values. Arreola-Risa and DeCroix [1] consider an (s, S) system subject to disruptions where only a fraction of the orders are backordered. They indicate that the dominant factor in this model is typically the relationship between the cost parameters. Krishnamoorthy and Raju [17] also consider an (s, S) system; while their models do not incorporate disruptions, they focus on alternate sourcing strategies if demand uncertainty leads to a need for supply that cannot be met in time by an original supplier and must be satisfied from local sources.

Parlar [21] evaluates a (Q, r) system with disrupted supply, random demand, and random lead times, and demonstrates the increase in cost from using solutions that ignore disruptions. Gurler and Parlar [16] expand this model to consider multiple suppliers that may each be disrupted. They indicate that a policy where different quantities are ordered based on the number of suppliers available would improve costs.

Using any state-dependent data available is commonly advocated. Song and Zipkin [35] examine ordering based on available system information. They model supply uncertainty using variable lead times and indicate that it is important to change the order quantity based on the quoted lead time at the time of the order. Tomlin and Snyder [34] advocate using available system information to improve ordering decisions. They model a periodic review system where advanced warnings of disruptions may be available. Lodree and Taskin [19] incorporate hurricane warnings into their disaster-planning inventory model, where they update stocking quantities based on updated hurricane wind speed data. Ross et al. [26] also include state-dependent information where both disruption and demand probabilities may change. They construct a continuous time Markov chain model for costs and show that using time-dependent solutions provides lower costs and added robustness.

Our terminology for describing strategies for coping with supply uncertainty is borrowed from Tomlin [37], who discusses three types of strategies: inventory control, sourcing, and acceptance. Inventory control strategies involve the ordering and stocking decisions and can be considered mitigating, proactive techniques. Sourcing strategies are contingency plans and can be reactive to an actual shortage or used proactively in planning for a potential shortage, and involve using back-up suppliers. Acceptance is used when the cost of coping with uncertainty outweighs the losses involved in passively accepting it. Tomlin [38] discusses the impact of supplier reliability forecasting on the decision of whether to source from a second supplier, and Tomlin and Wang [39] consider a combination of dual-sourcing and mix-flexibility to handle unreliable supply. Babich et al. [4] discuss pricing competition between multiple suppliers, all of whom have a disruption risk. They focus on a single uncertain demand and model the retailer's profits based on the number of suppliers they choose to order from. Babich et al. [3] model multiple suppliers with yield uncertainty and show how restrictions on investment funds can restrict the number of suppliers that would otherwise be optimal.

Chopra, Reinhardt, and Mohan [11] (hereafter referred to as CRM) consider a case where the primary supplier is unreliable and there exists a backup supplier who is perfectly reliable but more expensive. They consider deterministic demand and model the unreliable supplier with random yield and the possibility of a disruption. Their decision variables are the order quantity from the unreliable supplier and the quantity to reserve from the second supplier. This reservation quantity is the maximum amount that can be ordered in the case of a supply shortage, and has an associated per-unit cost. Their analysis considers only a single-period model, assuming that inventory and backorders are not carried period-to-period.

CRM develop closed-form solutions for the optimal order quantity and reservation quantity. They then compare these results to the quantities which would be ordered if the inventory manager bundled the variance (i.e., assumed that the periods in which zero items are received are governed by the yield uncertainty process rather than the disruption process), and demonstrate the cost increases that arise. They conclude that bundling the variance causes over-ordering from the primary supplier and under-utilization of the secondary supplier.

We consider an inventory system that faces both yield uncertainty and the risk of complete supply disruptions simultaneously. Unlike CRM, we consider a multi-period setting to take advantage of inventory as both a proactive and a reactive tool. We consider both single-supplier and two-supplier models and compare optimal order policies to those found using single-period models. We show that if the newsboy fractile and/or disruption probability are high, using a single-period approximation causes under-ordering from the primary supplier. We demonstrate that single-period time truncation distorts the optimal order quantities, increases the cost, and may lead to the wrong overall strategy for supply risk mitigation.

3 Single-Supplier Model

3.1 Assumptions

We consider a firm that procures product from a single supplier subject to yield uncertainty and randomly disrupted supply. The firm operates under a periodic-review base-stock policy, with s representing the base-stock level. The sequence of events in each period is as follows: the firm observes the current inventory level (IL), places an order of size $s-\text{IL}$, either receives supply (bringing the inventory level to a value y) or receives nothing (in a disrupted period), realizes demand, and incurs costs.

We assume zero lead time and deterministic demand (d per period), and that unmet demands are backordered. The base-stock level, s, is our decision variable. We use π_i to denote the steady-state probability that the system has been disrupted for i consecutive periods, $i \in [0, \infty)$, and assume that disruptions last an integer number of periods. The π_i form the pdf of the disruption process. They may arise from a simple discrete-time Markov process with *on* and *off* states (as in, e.g., [20, 25, 27, 33]), or from a more general supply process (as in, e.g., [37]). Since an i-period disruption cannot occur unless an $(i-1)$-period disruption has already occurred, $\pi_i \leq \pi_{i-1}$ for all $i \geq 1$, but our mathematical results do not require this.

During non-disrupted periods, the yield has an additive random quantity w. We assume w is generally distributed with mean $\overline{w}$ and is independent of the order quantity. Thus $y = s + w$. In some applications (e.g., defective products) w is always non-positive, while in others (e.g., batch processing of chemicals), it may be positive or negative. Our model accommodates either case. We use the notation $f(\cdot)$, $F(\cdot)$, and $G(\cdot)$ to denote the pdf, cdf, and loss function of w, and $f_y(\cdot)$, $F_y(\cdot)$, and $G_y(\cdot)$ to denote the same for y. We assume the probability of a negative delivery from the supplier is negligible ($F(-s) \approx 0$), as is the probability of receiving more than d plus the base-stock level ($1 - F(d) \approx 0$) so that an order will be placed in every period; however, we use $-\infty$ and ∞ as the upper and lower limits for integrals involving w.

The cost parameters are given by C_o, the overage cost per unit per period, and C_u, the backorder cost per unit per period. Typically $C_o < C_u$, but this is not required for analysis. We refer to $\frac{C_u}{C_o+C_u}$ as the newsboy fractile.

We first analyze the exact cost function and discuss its solution, then discuss an approximate method for evaluating the cost: single-period truncating. We assume that no other strategies are available for mitigating the supply uncertainty (no backup supplier is available), thus acceptance and inventory control are the only recourse options. We relax this assumption in Section 4.

3.2 Analysis of the Single-Supplier Model

3.2.1 System Costs

A successful period (one without a disruption) incurs the following expected cost:

$$C_o \int_d^{\infty} (y-d) f_y(y) dy + C_u \int_{-\infty}^{d} (d-y) f_y(y) dy \tag{1}$$

For the last in a string of $i-1$ successive disrupted periods, $i \geq 1$, the firm will have observed id total units of demand since the last successful delivery brought the inventory level to y. Thus the expected cost in such a period is:

$$C_o \int_{id}^{\infty} (y-id) f_y(y) dy + C_u \int_{-\infty}^{id} (id-y) f_y(y) dy \tag{2}$$

Essentially after every period we have newsboy-like costs centered around id, but with the variability coming from supply instead of demand. Then the total expected cost is:

$$C_{1s}(s) = \sum_{i=1}^{\infty} \left[\pi_{i-1} \left(C_o \int_{id}^{\infty} (y - id) f_y(y) dy + C_u \int_{-\infty}^{id} (id - y) f_y(y) dy \right) \right] \tag{3}$$

(The subscript $1s$ indicates "one supplier".) Since $y = s + w$, we can rewrite the expected cost as:

$$C_{1s}(s) = \sum_{i=1}^{\infty} \left[\pi_{i-1} \left(C_o \int_{id-s}^{\infty} (s + w - id) f(w) dw + C_u \int_{-\infty}^{id-s} (id - s - w) f(w) dw \right) \right] \tag{4}$$

This can be reduced to the form in the following proposition.

Proposition 1 *The expected cost for a retailer that has additive yield uncertainty and is subject to disruptions is:*

$$C_{1s}(s) = -C_u(s + \overline{w}) + \sum_{i=1}^{\infty} \left[\pi_{i-1} \left(C_u id + (C_o + C_u) G(id - s) \right) \right] \tag{5}$$

where $G(x) = \int_x^{\infty} (t - x) f(t) dt$ *represents the loss function for* w.

Proof: See the Appendix, Section A.1.

The next proposition provides the first-order condition for the s that minimizes $C_{1s}(s)$.

Proposition 2 $C_{1s}(s)$ *is convex, and the* s *that minimizes* $C_{1s}(s)$ *satisfies*

$$C_o - (C_o + C_u) \sum_{i=1}^{\infty} \left[\pi_{i-1} F(id - s) \right] = 0 \tag{6}$$

Proof: See the Appendix, Section A.2.

Since s appears in the argument to $F(\cdot)$ in an infinite number of terms, each time with a different multiple of d, there is no known closed-form solution to the first-order condition, though Schmitt et al. [27] provide an approximation that can be solved in closed form when w is normally distributed. To find s^* exactly, we must use line-search techniques; this is the approach we take in our computational studies.

3.2.2 Special Cases

We can use Proposition 2 to find a closed-form expression for s^* in the special case in which there is yield uncertainty but no disruptions (i, the number of consecutive disruptions, is always equal to 0):

$$\begin{aligned} \frac{d}{ds} C_{1s,i=0}(s) &= C_o - (C_o + C_u) F(d - s) = 0 \\ \Rightarrow s^*_{i=0} &= d - F^{-1} \left(\frac{C_o}{C_o + C_u} \right) \end{aligned} \tag{7}$$

Unless $\overline{w} > 0$ or $C_o > C_u$ (both of which are somewhat unusual), we will have $F^{-1}\left(\frac{C_o}{C_o+C_u}\right) < 0$. Thus the firm operates with a base-stock level which is equal to the demand plus a positive quantity that depends on the cost and yield parameters, paralleling the newsboy safety stock concept under demand uncertainty. This significance of the newsboy fractile in supply uncertainty models has been discussed by Dada et al. [12] for multiple suppliers and by Tomlin [37] and Schmitt et al. [27] for supply disruptions.

It is also interesting to consider the case in which there is no yield uncertainty but there may be supply disruptions. In this case, the optimal base-stock level is an integer multiple of d [37]. Then we can write the cost as follows. (We use $a^+ \equiv \max\{a, 0\}$ and $a^- \equiv \max\{-a, 0\}$.)

$$C_{1s,w=0}(s) = \sum_{i=1}^{\infty} \pi_{i-1}(C_o(s - id)^+ + C_u(id - s)^+) \tag{8}$$

In this case, the optimal solution is given by

$$s^* = j^*d \text{ where } j^* \text{ is the smallest } j \geq 1 \text{ such that } \sum_{i=1}^{j} \pi_{i-1} \geq \frac{C_u}{C_u + C_o}. \tag{9}$$

(This is a special case of a result by Tomlin [37]. See also Schmitt et al. [27].) Thus the newsboy fractile, $\frac{C_u}{C_u+C_o}$, is the service level for the system. The optimal base-stock level is an increasing step function of π_i and the newsboy fractile, with the steps decreasing in width as each increases. As the disruption risk or required service level increase, the optimal solution is to stock additional periods' worth of demand.

3.3 Truncating the Time Periods

We now evaluate an approximation which truncates the time horizon to a single period. Unmet demands are lost and excess inventory has no value. One can show that the optimal solution in this case is

$$s_t^* = d - F^{-1}\left(\frac{C_o}{C_o + C_u}\right). \tag{10}$$

(CRM prove this in an earlier draft of their paper [10]). The right-hand side of (10) is equal to that of (7), which we likened to a single-period newsboy solution. This is a special case of the general model presented by Dada et al. [12].

Note that s_t^* is independent of disruption risk, which seems counter-intuitive. Since extra inventory is not carried into the future in this system, over-ordering will not help mitigate future

Table 1: Single-Supplier Model Parameters

Variable	Value
d (demand)	100
σ (supply st.dev.)	4
α (disruption probability)	0.02
β (recovery probability)	0.5
$\frac{C_u}{C_o+C_u}$ (newsboy fractile)	0.95

disruptions. Only yield uncertainty can be mitigated in this system. Essentially the acceptance strategy is being used here to handle disruptions. This is the crux of the error that results from using a single-period model for inventory problems with disruptions.

3.4 Numerical Results: Single-Supplier System

In the sections that follow, we explore the impact of varying $\frac{C_u}{C_u+C_o}$ and disruption parameters on the optimal solution and its expected cost. We must first specify a distribution for disruptions and yield. We assume disruptions follow a discrete-time Markov chain with 2 states, disrupted and non-disrupted. The transition probability from the non-disrupted to the disrupted state is given by α (called the disruption probability), and that from the disrupted to the non-disrupted state by β (the recovery probability). This allows us to test the impact of both the frequency and the duration of disruptions. Following CRM, we assume the yield is normally distributed with mean 0 and variance σ^2. Note that this means the quantity received from the supplier may exceed that which was ordered, but this assumption is not unreasonable; for example, CRM suggest this distribution would be valid in cases where contracts are based on production starts but yield is random, such as in flu vaccine or semiconductor manufacturing [11].

We use the parameter values given in Table 1 as a base setting.

3.4.1 Impact of the Newsboy Fractile

We first evaluate the optimal base-stock level as the newsboy fractile varies. The special cases presented in Section 3.2.2 demonstrate that this fractile acts as a service level requirement. We fixed $C_o = 10$ and varied C_u. We optimized (5) numerically and constructed Figure 1(a) to demonstrate the optimal base-stock level for different values of $\frac{C_u}{C_u+C_o}$. This graph demonstrates that the principles discussed in Section 3.2.2 regarding equation (9) still hold when yield uncertainty is present. Each sharp vertical climb in the optimal solution curve occurs when j must increase by 1 in order to maintain $\sum_{i=1}^{j} \pi_{i-1} \geq \frac{C_u}{C_u+C_o}$. These climbs do not move among exactly integer

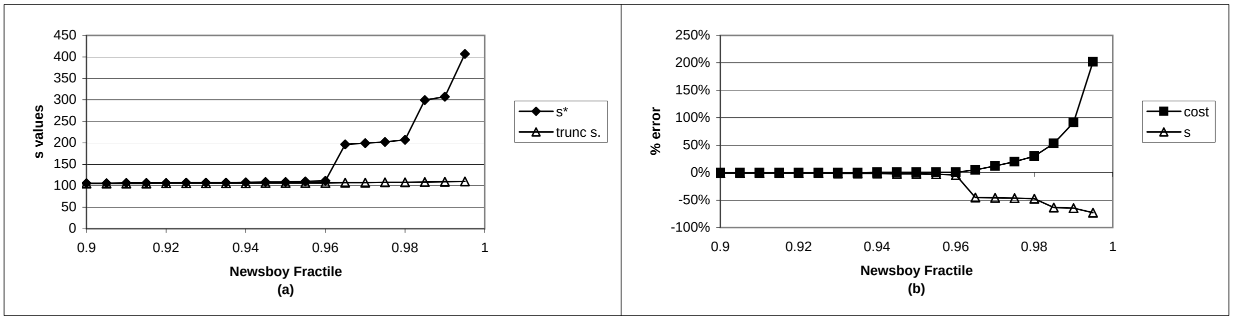

Figure 1: (a) 1-supplier Model s Solutions and (b) Truncated Solution Error versus $\frac{C_u}{C_u+C_o}$

multiples of the demand; the stochasticity of the yield causes the optimal base-stock level to be slightly below an integer number of periods of demand for lower $\frac{C_u}{C_u+C_o}$ and become slightly above that integer number of periods of demand as $\frac{C_u}{C_u+C_o}$ approaches the next jump point. Figure 1(a) clearly demonstrates that as the cost of stockouts increases, the optimal inventory strategy involves stocking extra periods of demand to mitigate the risk of stockouts from disruptions.

Figure 1(a) also includes the truncated model's base-stock level, s_t^*. The truncating approximation fails to recognize the need to stock multiple periods of demand for high newsboy fractiles, because it stocks for the current period only and ignores future disruptions. It consistently underestimates the required base-stock level. We present the error that the truncated model provides in terms of increased cost and understocked inventory in Figure 1(b). For newsboy fractile values of 99% and 99.5% the cost increases from the truncated model are 91% and 202%, respectively, demonstrating the importance of accounting for future periods.

3.4.2 Impact of Supply Parameters

We also varied the supply parameters to view their effects on the optimal and truncated solution. We found that changes in σ did not have a large impact on the differences between the models. Both models increase s to add more safety stock as σ increases, and the relative cost error (depending on the newsboy fractile or disruption parameters) remains roughly the same. Due to space considerations, we omit detailed results here.

The disruption and recovery probabilities, α and β, do have a large impact on the difference between the two models. We set the newsboy fractile equal to 95% ($C_u = 190$), and varied α and β. Figure 2 illustrates the impact of α on the solutions, and Figure 3 illustrates the impact of β.

The error that results from using a truncated model is clear from both figures. In Figure 2, when the disruption probability increases, the exact model stocks multiple periods of demand, for

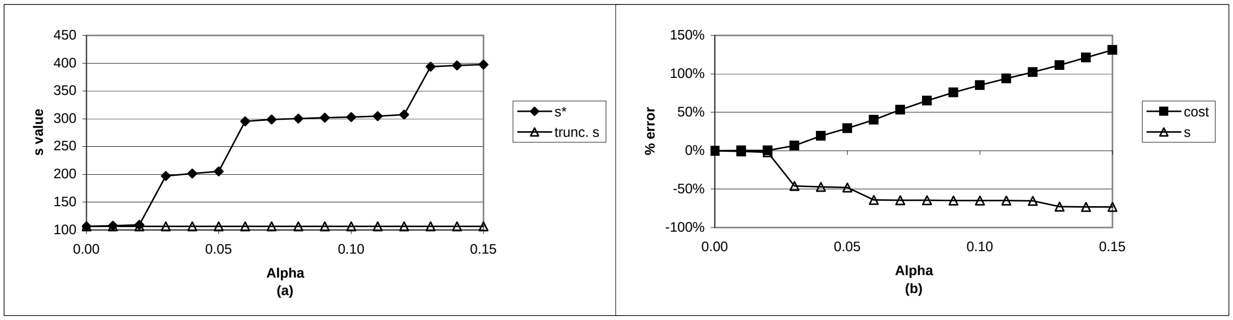

Figure 2: (a) 1-supplier Model s Solutions and (b) Truncated Solution Error versus α

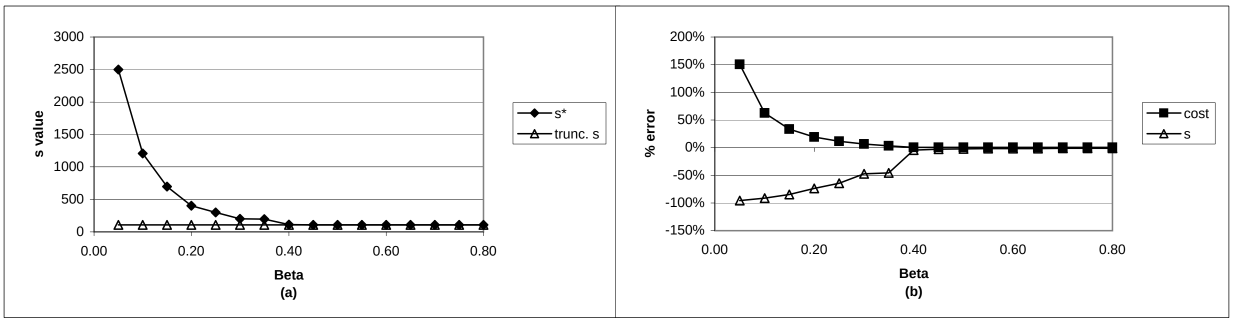

Figure 3: (a) 1-supplier Model s Solutions and (b) Truncated Solution Error versus β

the same reasons that it does when the newsboy fractile increases, as in Figure 1. The truncated model never allows for the switch in strategy, in which significant inventory should be held in order to mitigate the disruption risk. Similarly, when the recovery probability is very small, the truncated model significantly underestimates the necessary base-stock level, even when the disruption probability is also small ($\alpha = 0.02$). For example, when β is 5%, s_t is 96% smaller than s^*. Thus the long-term risks posed by disruptions are not adequately captured by the truncated model; full inventory back-up is clearly optimal if the disruption risk is significant enough, and the truncated model never chooses that strategy.

4 Two Supplier Model

4.1 Assumptions

Suppose now that the firm has a second supplier available who is perfectly reliable; it always delivers exactly what is ordered. However, capacity must be reserved at this reliable supplier, and the firm cannot order more than it reserves. The firm pays r per unit reserved (regardless of whether it actually orders these units), and then an additional cost p_2 per unit actually ordered. We also assume the firm pays p_1 per unit received from the first supplier. (We ignored this cost in the single-supplier model since the mean number of units ordered per period, and hence the mean ordering cost per period, is a constant, independent of s.) The total number of units that the firm reserves per period is denoted R and is a decision variable, as is s, the base-stock level for the primary supplier.

To avoid trivial instances, we assume that $p_2 + r > p_1$, meaning it is less expensive to buy a product from the primary (though unreliable) supplier. Also, since we assume that supply can never be negative and demand is deterministic, we have the following intuitive result, whose proof we omit.

Lemma 3 *If demand is deterministic and the second supplier is perfectly reliable, then* $0 \leq R^* \leq d$, *where* R^* *is the optimal reservation quantity from the second supplier.*

Since the second supplier is perfectly reliable, there would never be a need to reserve more than d from them. Although it may be beneficial for R to change from period to period to reflect current inventory levels and disruption status, we assume instead that the retailer chooses R at the start of the horizon and may not change it subsequently. This assumption is reasonable for a case where the relationship with the second supplier is contractual and terms must be fixed in advance, or where the second supplier cannot be flexible with their capacity allocation.

The sequence of events in each period is as follows: the firm observes the current inventory level (IL_{2s}). It places an order of size $s - IL_{2s}$ from the primary supplier and either receives supply (bringing the inventory level to y) or receives nothing (in a disrupted period); in either case, the resulting IL_{2s} is denoted x. Then if $x < d$, it orders $\min\{R, d - x\}$ from the second supplier. We assume that the firm never orders more than $d - x$ from the second supplier (that is, it does not place "proactive" orders from supplier 2). Finally, it realizes demand and incurs costs.

As in Section 3, we first analyze the exact cost function and discuss its solution, then discuss the truncating heuristic.

4.2 Analysis of the Two-Supplier Model

4.2.1 Expected Inventory Levels

The expected inventory level for the two-supplier model, $E[IL_{2s}]$, is used to determine the expected order quantities from the two suppliers, as well as the expected holding and stockout costs. Proposition 4 presents the expectation of the positive and negative parts of $E[IL_{2s}]$.

Proposition 4 *The expected positive and negative parts of the starting inventory level for the two supplier system are given by:*

$$E[IL_{2s}^+] = \sum_{j=1}^{\infty} \left[\pi_{j-1} \int_{jd}^{\infty} (y - jd) f_y(y) dy \right] \quad (11)$$

$$\begin{aligned} E[IL_{2s}^-] \quad = \quad & \left[\sum_{j=1}^{\infty} \left[\pi_{j-1} \int_{-\infty}^{d-R} (j(d-R) - y) f_y(y) dy \right] + \right. \\ & (1-\alpha) \sum_{j=2}^{\infty} \sum_{k=1}^{j-1} \left[\alpha^{j-1} \left(\int_{kd-R}^{kd} (j-k)(d-R) f_y(y) dy \right. \right. + \\ & \left. \left. \left. \int_{kd}^{(k+1)d-R} (jd - (j-k)R - y) f_y(y) dy \right) \right] \right] \end{aligned} \quad (12)$$

Proof: See the Appendix, Section A.3.

4.2.2 Supplier Order Quantities

In order to model the expected costs, we next characterize the order quantities from each supplier. Since the firm orders $s - IL_{2s}$ from supplier 1, the expected order quantity for supplier 1 is $s - E[IL_{2s}]$, where $E[IL_{2s}] = E[IL_{2s}^+] - E[IL_{2s}^-]$. (The latter quantities are obtained from Proposition 4.)

Next consider the second supplier. Recall that y is the inventory level after the last successful delivery from the first supplier. Suppose the first supplier has been disrupted for $(j-1)$ periods, $j \geq 1$. (If $j = 1$, then the supplier has experienced 0 consecutive disruptions; i.e., is in a successful delivery period). If $y \leq jd - R$, then the firm orders R from supplier 2. If $jd - R < y < jd$, it orders the difference, $jd - y$, and if $y \geq jd$, it orders nothing. Thus the expected quantity ordered in a given period from the second supplier, given that the period is the $(j-1)^{st}$ consecutive disrupted period, is:

$$\int_{-\infty}^{jd-R} R f_y(y) dy + \int_{jd-R}^{jd} (jd - y) f_y(y) dy \quad (13)$$

4.2.3 Expected Cost

The expected holding and penalty costs are $C_o E[IL_{2s}]^+$ and $C_u E[IL_{2s}]^-$, respectively. Thus we have the following total expected cost:

$$\begin{aligned} C_{2s}(s, R) \quad = \quad & \sum_{i=1}^{\infty} \pi_{i-1} p_2 \left(\int_{-\infty}^{id-R} R f_y(y) dy + \int_{id-R}^{id} (id - y) f_y(y) dy \right) + \\ & C_o E[IL_{2s}^+] + C_u E[IL_{2s}^-] + \pi_0 \; p_1 \left(s - E[IL_{2s}] \right) + rR, \end{aligned} \quad (14)$$

where $E[IL^+_{2s}]$ and $E[IL^-_{2s}]$ are as given in Proposition 4. This is equivalent to the form given in the following proposition.

Proposition 5 *The expected cost for the two supplier model is given by:*

$$\begin{aligned} C_{2s}(s,R) &= (p_2+r)R + p_2\sum_{i=1}^{\infty}\left[\pi_{i-1}\left(G(id-s)-G(id-R-s)\right)\right] + \\ &\quad C_o E[IL^+_{2s}] + C_u E[IL^-_{2s}] + \pi_0 p_1\left(s - E[IL^+_{2s}] + E[IL^-_{2s}]\right) \end{aligned} \tag{15}$$

where

$$E[IL^+_{2s}] = \sum_{j=1}^{\infty}\left[\pi_{j-1}G(jd-s)\right] \tag{16}$$

and

$$\begin{aligned} E[IL^-_{2s}] &= d-R-s-\overline{w}+G(d-R-s)+(d-R)F(d-R-s)\sum_{j=1}^{\infty}\left[\pi_{j-1}(j-1)\right] + \\ &\quad \sum_{j=2}^{\infty}\sum_{k=1}^{j-1}\Big[\pi_{j-1}\Big(G((k+1)d-R-s)-G(kd-s)+ \\ &\quad (d-R)(1-(j-k)F(kd-R-s)+(j-k-1)F((k+1)d-R-s))\Big)\Big] \end{aligned} \tag{17}$$

Proof: See the Appendix, Section A.4.

We minimize (15) in our numerical studies to solve for the optimal s^* and R^*. Numerical evaluation of the second derivatives of (15) using normally distributed yield showed that it is not always convex, but for most instances, it is quasiconvex; that is, that there exists a single minimum-cost solution. For the results presented in Section 4.4, we use *What'sBest!* Software, a LINDO Systems program for Excel, which has a global solver option. The global solver partitions the problem into multiple convex sub-problems and uses branch-and-bound heuristics to exhaustively evaluate the objective function and find the global optimal solution. Therefore, even for instances that are not quasiconvex, the solutions in Section 4.4 are guaranteed to be optimal.

4.3 Truncating the Time Periods

CRM conducted extensive analysis on the truncated version of the 2-supplier system and established closed-form solutions for the optimal s and R in the case where we consider only one period. We include a purchase cost at the first supplier where they do not, but otherwise consider the same system. The expected cost for the truncated system is given in the following proposition.

Proposition 6 (Chopra et al., 2007) *The expected cost for the truncated two-supplier system is:*

$$C_{2s,t}(s,R) = (r+p_2)R + C_u(d-R) + (1-\alpha)\left[C_u(G(d-R-s)-s-\overline{w})+ p_2(G(d-s)-G(d-R-s)) + C_oG(d-s) + p_1(s+\overline{w})\right] \quad (18)$$

Proof: Follows immediately from CRM, equation (7), however our function also includes the average expected cost from the primary supplier of $p_1(s+\overline{w})$.

From (18) we can derive the optimal order and reservation quantities for the truncated system. CRM proved that their cost function (equivalent to (18) excepting the p_1 term) is convex, and therefore (18) is convex as well.

Proposition 7 *The optimal reservation and order quantities for the truncated system are as follows:*

$$R_t^* = \max\left\{0, F^{-1}\left(\frac{\alpha(C_u-p_2)-r+(1-\alpha)(C_o+p_1)}{(1-\alpha)(C_o+p_2)}\right) - F^{-1}\left(\frac{r-\alpha(C_u-p_2)}{(1-\alpha)(C_u-p_2)}\right)\right\} \quad (19)$$

$$s_t^* = \begin{cases} d - F^{-1}\left(\frac{\alpha(C_u-p_2)-r+(1-\alpha)(C_o+p_1)}{(1-\alpha)(C_o+p_2)}\right), & \text{if } R_t^* > 0; \\ d - F^{-1}\left(\frac{C_o+p_1}{C_o+C_u}\right), & \text{if } R_t^* = 0. \end{cases} \quad (20)$$

Proof: See the Appendix, Section A.5.

These are identical to the results found by CRM, with two exceptions. One is the inclusion of the p_1 term. The second is the expression for s_t^* in the case when $R^* = 0$, which CRM omit. Since s_t^* does not depend on α if $R^* = 0$, this indicates that if the disruption risk does not warrant use of the second supplier, then the optimal truncated solution is to ignore disruptions. This parallels the results found in the single-supplier truncated model.

In order for these equations to be well defined, the arguments of $F^{-1}(\cdot)$ must be strictly between 0 and 1. This may be violated if α, C_u, p_2, or r are large, and it is always violated if $C_u < p_2$. If $C_u < p_2$, the truncated model would always choose to stock out for a single period rather than spend more to purchase material from supplier 2. If either of these violations occurred, then the closed-form optimal solutions given in Proposition 7 could not be used, and (18) would require numerical optimization instead.

4.4 Numerical Results: Two-Supplier System

The base parameter values used are given in Table 2. As in Section 3, we determined the optimal solution for s and R numerically. We again assume that disruptions occur with probability α,

Table 2: Two-Supplier Model Parameters

Parameter	Value
d (demand)	100
σ (supply st.dev.)	4
α (disruption probability)	0.02
β (recovery probability)	0.5
$\frac{C_u}{C_o+C_u}$ (newsboy fractile)	0.95
p_1 (supplier 1 per-unit cost)	10
p_2 (supplier 2 per-unit cost)	15
r (reserve price)	5

recoveries from a disruption occur with probability β, and the random yield is normally distributed with mean 0 and variance σ^2.

For the truncated solution, we used the equations in Section 4.3. We then evaluated the truncated solutions using the exact expected cost function for comparison. For testing ranges of inputs, as noted in Section 4.3, the truncated solution equations are not well defined for all parameter values (for example, $\alpha > 0.05$). We chose parameter values for which the model is well defined.

4.4.1 Impact of the Newsboy Fractile

Figure 4 shows the optimal order and reservation quantities and cost comparisons for the two-supplier model when the newsboy fractile varies. An important observation for these results is that as the newsboy fractile increases, the optimal solution undergoes a jump; when $\frac{C_u}{C_o+C_u} = 93.5\%$, the reservation quantity increases to cover almost all of the demand. This mimics the case in the single-supplier model; in that system, when $\frac{C_u}{C_o+C_u}$ was high enough, the optimal base-stock level jumped to equal roughly two periods of demand. Here, the jump again indicates that the overall strategy undergoes a qualitative change. The optimal strategy at $\frac{C_u}{C_o+C_u} = 93\%$ is to reserve enough to cover mainly yield uncertainty, but at 93.5% it is to cover the disruption risk as well.

The results suggest that, in the truncated model, for large $\frac{C_u}{C_o+C_u}$ values the costs are significantly higher. The increase in cost as $\frac{C_u}{C_o+C_u}$ increases is similar to that seen in the one-supplier results. We note that the truncated system underutilizes both the primary and the secondary supplier for all values of $\frac{C_u}{C_o+C_u}$. Thus the truncated system underestimates the impact of high-cost disruptions and does not sufficiently provide protection against them.

The truncating method fails to make the jump to reserving the full demand quantity because it underestimates the impact of disruption risk. This ability to correctly select the optimal strategy

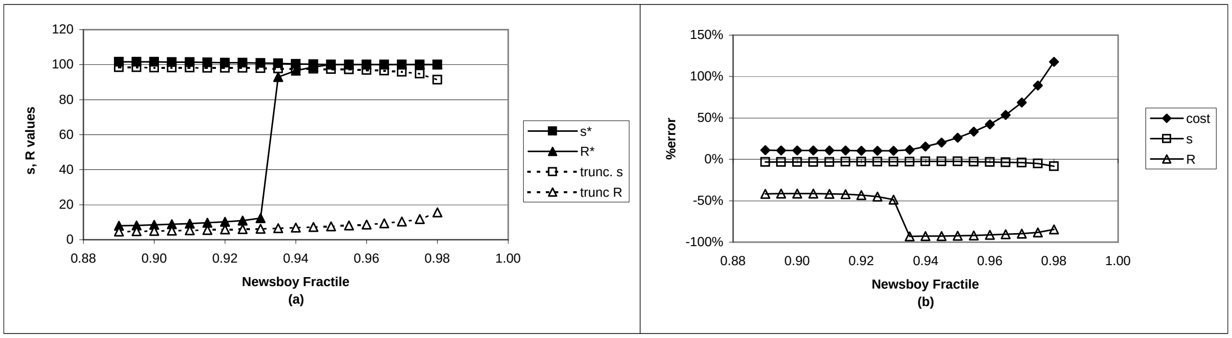

Figure 4: (a) 2-Supplier Model s and R and (b) Truncated Solution Error versus $\frac{C_u}{C_o+C_u}$

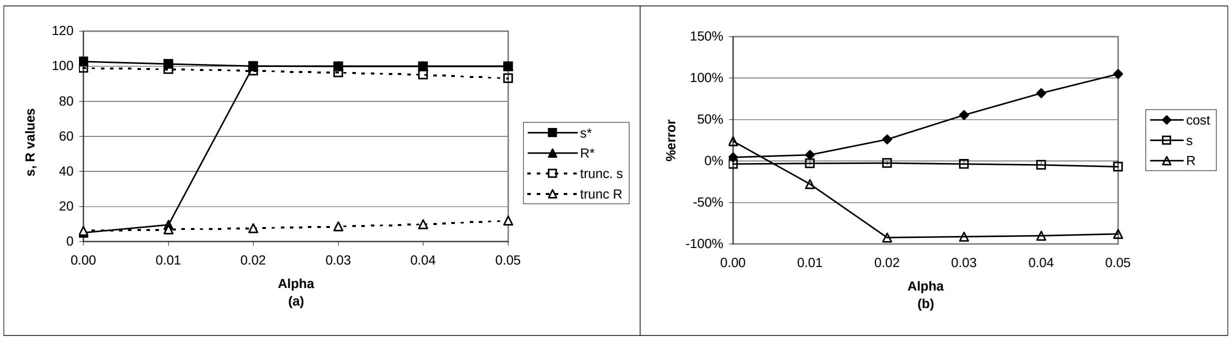

Figure 5: (a) 2-Supplier Model s and R and (b) Truncated Solution Error versus α

and provide full supplier redundancy is a strong advantage of the exact, infinite-horizon model over the truncated model.

4.4.2 Impact of Supply Parameters

We also evaluated the system while varying the supply parameters, setting $C_u = 190$, or $\frac{C_u}{C_o+C_u} = 0.95$. (Larger $\frac{C_u}{C_o+C_u}$ and α values combined tend to make the truncated solution not be well defined). As in Section 3.4.2, σ did not have a large impact on the relative difference between the models; both models increase R to provide more protection against yield uncertainty, but not enough to cause a large change in the cost error. (Detailed results are omitted.) Both α and β again have a large impact on the solutions, and graphs of the first-supplier base-stock levels and the second-supplier reserve quantities and cost comparisons can be found in Figures 5 and 6.

As α increases, the optimal strategy shifts from reserving very little from the second supplier (mainly to protect against yield uncertainty) to ordering the full demand quantity (to protect against disruptions). This is similar to the order quantity in the single-supplier model. Similarly, as β decreases, it becomes optimal to reserve the full demand quantity from the second supplier in order to ensure that the firm can outlast a disruption that is multiple periods long. We found that this strategic move to back up the full demand quantity also occurred when the reservation

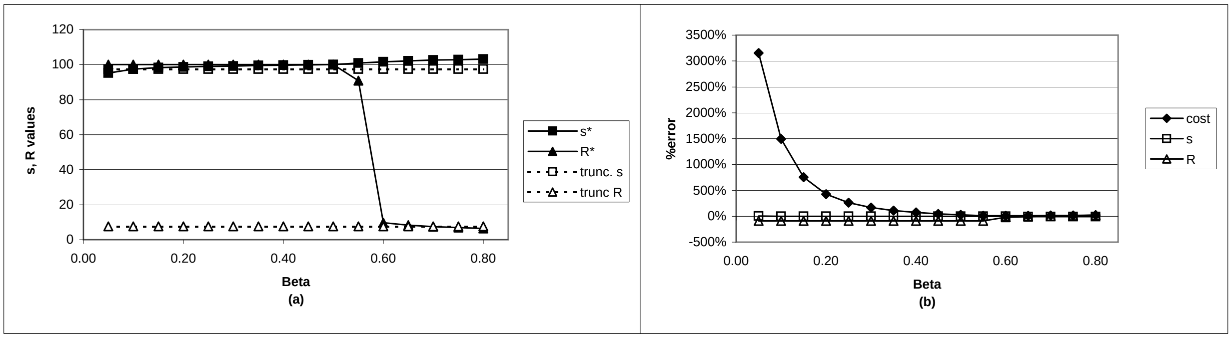

Figure 6: (a) 2-Supplier Model s and R and (b) Truncated Solution Error versus β

price, r, was very small. The truncated model always fails to recognize the potential need for this strategy because it always underestimates the long-term impact of disruptions. It does not sufficiently mitigate the risks posed by the unreliable supplier.

4.4.3 Discussion of Results

Completely ignoring disruptions has a similar impact as using the truncated model. For example, if we set $\alpha = 0$ and let all inputs remain at their base levels given in Table 2, the system would set $s = 103$ and $R = 5$. Since the disruption probability $\alpha = 0.02$ in the base system and the true optimal solution is $s^* = R^* = 100$, ignoring disruptions in this case would cause a cost increase of 19%. Clearly, the need to capture the impact of supply disruptions on the system is significant.

Both the single-supplier and two-supplier models capture the need to provide full redundancy if disruption risk or newsboy fractiles are high. The single-supplier model moves to stocking multiple periods' worth of material if the risk is high enough, and similarly, the two-supplier model moves to having multiple periods' worth of material available (in the form of the second supplier). These choices involve significant up-front costs, but provide significant protection to the system if a disruption should occur.

5 Conclusions

In order to smoothly and efficiently mitigate supply chain risk, sourcing and inventory strategies should proactively take that risk into account. The analysis presented in this paper demonstrates the complexity of systems with uncertain supply when that supply is subject to both yield variability and disruptions, and the importance of planning for future periods by considering more than a single-period model. From our numerical studies, we draw the following conclusions:

- Truncating the Time Periods
  - For high newsboy fractiles (or required service levels), moderate disruption probabilities, or low recovery probabilities, using the single-period model underestimates the impact of disruption risk, causing a significant increase in cost.
  - Truncating generally underutilizes the unreliable supplier in both models and the reliable supplier in the two-supplier model.
- Overall Strategies
  - When disruption risk is serious enough (when the penalty costs or frequencies are high or durations are long), the optimal inventory strategy is often to stock multiple periods of demand (either by ordering multiple periods of demand from the unreliable supplier in the single-supplier system, or through significant backup from the reliable supplier in the two-supplier system).
  - The single-period model fails to recognize this switch of optimal overall strategy and never selects full system inventory protection or supplier redundancy.

The results clearly demonstrate the importance of analyzing and mitigating disruption risk proactively through planning for future disruptions. Use of a truncated model or ignoring disruptions entirely leaves the system at risk for significant material shortage. While yield variability requires safety stock to avoid stockouts, disruptions can require either significantly more safety stock or the use of a backup supplier when available. If disruptions can last for multiple periods, and proactive mitigation through inventory and/or backup suppliers is possible, then multiple-period models should be used for making strategic decisions.

Potential areas for future research include allowing demand to be stochastic, exploring the two-supplier case when both suppliers are unreliable, or including a fixed cost for the unreliable supplier that might drive using the more expensive supplier as the single source. Also, the strategies modeled here could be examined for their downstream impact in multi-echelon settings. Real occurrences in the business world have demonstrated that variable and disrupted supply can have a significant impact on supply chains, and it is important that policies for dealing with these risks continue to be explored.

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A Appendix: Proofs of Propositions

We use several properties of the loss function, so we introduce the following lemma for reference:

Lemma 8

Where $f(t)$ *and* $F(t)$ *are the pdf and cdf for a general distribution with mean* $\bar{t}$, $G(x) = \int_x^\infty (t - x)f(t)dt = \int_x^\infty (1 - F(t))(t)dt$ *is the loss function, and* $h(t)$ *is any function of* t*, we have:*

$$\int_{-\infty}^{x} (t-x)f(t)dt = \bar{t} - x - G(x) \tag{21}$$

$$\frac{d}{dt}G(h(t)) = -h'(t)[1 - F(h(t))] \tag{22}$$

$$\int_{-\infty}^{x} tf(t)dt = \bar{t} - x(1 - F(x)) - G(x) \tag{23}$$

Proof: These follow immediately from standard sources, e.g. [2], [9], [41].

A.1 Proof of Proposition 1, Section 3.2.1

From (4),

$$C_{1s}(s) = \sum_{i=1}^{\infty} \left[\pi_{i-1} \left(C_o \int_{id-s}^{\infty} (s + w - id)f(w)dw + C_u \int_{-\infty}^{id-s} (id - s - w)f(w)dw \right) \right] \tag{24}$$

Applying the loss function and (21) we have

$$\begin{aligned}
C_{1s}(s) &= \sum_{i=1}^{\infty} \left[\pi_{i-1} \left(C_o \int_{id-s}^{\infty} (s + w - id)f(w)dw + C_u \int_{-\infty}^{id-s} (id - s - w)f(w)dw \right) \right] \\
&= \sum_{i=1}^{\infty} \left[\pi_{i-1} \left(C_o \int_{id-s}^{\infty} (w - (id - s))f(w)dw - C_u \int_{-\infty}^{id-s} (w - (id - s))f(w)dw \right) \right] \\
&= \sum_{i=1}^{\infty} \left[\pi_{i-1} \left((C_o + C_u)G(id - s) - C_u(\overline{w} - id + s) \right) \right] \\
&= -C_u(\overline{w} + s) + \sum_{i=1}^{\infty} \left[\pi_{i-1} \left(C_u id + (C_o + C_u)G(id - s) \right) \right]
\end{aligned} \tag{25}$$

□

A.2 Proof of Proposition 2, Section 3.2.1

The optimal s is found by setting $\frac{d}{ds}C_{1s}(s)$ equal to zero. Using (22) from Lemma 8, we have:

$$\begin{aligned}
\frac{d}{ds}C_{1s}(s) &= -C_u + \sum_{i=1}^{\infty} \left[\pi_{i-1} \left(-(C_o + C_u)(1 - F(id - s)) \right) \right] \\
&= C_o - (C_o + C_u) \sum_{i=1}^{\infty} \left[\pi_{i-1} F(id - s) \right]
\end{aligned} \tag{26}$$

Table 3: Inventory Level Cases for the Two-Supplier Model*

Case	Range for y	Description
1a	$y \geq d$	y exceeds demand
1b	$y < d$	y is short of demand
2a	$y > jd$	y exceeds current demand
2b	$y < d - R$	y was short of the last successful period's demand and reserve
2c	$y \in (kd, (k+1)d - R)$	y was short of the k^{th} failure-period's demand and reserve
2d	$y \in (kd - R, kd)$	y was short of the $(k-1)^{st}$ failure-period's demand, but not reserve

**Note:* $j-1$ *is the number of consecutive disrupted periods the firm has currently experienced (possibly equal to 0) and* $k-1$ *is the number of consecutively disrupted periods for which the ending* IL_{2s} *was* ≥ 0 $(k < j)$.

We prove the convexity of $C_{1s}(s)$ by confirming that its second derivative with respect to s is positive:

$$\frac{d^2}{ds^2}C_{1s}(s) = (C_o + C_u)\sum_{i=1}^{\infty}[\pi_{i-1}f(id - s)] \tag{27}$$

Since every term in the right hand side is positive, we conclude that $C_{1s}(s)$ is convex.

□

A.3 Proof of Proposition 4, Section 4.2.1

In order to establish the expected ending inventory level, we consider different cases for what the inventory level could be prior to deciding whether to order from Supplier 2. In a successful period (where no disruption occurs at Supplier 1), Case 1, this occurs following a delivery from Supplier 1. In a disrupted period, Case 2, this occurs at the beginning of the period. Table 3 outlines the cases that we consider for the inventory level immediately following the last successful delivery from Supplier 1, y. For each case, let $j-1$ be the number of consecutive disrupted periods the firm has currently experienced (possibly equal to 0) and $k-1$ be the number of consecutively disrupted periods for which the ending IL_{2s} was ≥ 0 $(k < j)$.

A.3.1 Case 1: Successful Delivery Period

In a successful period (where supply is not disrupted, though yield uncertainty can still cause stockouts), we have two cases for y. In Case 1a, $y \geq d$; the firm receives more supply than demand. In this case, IL_{2s} is simply the excess amount, $y - d$.

In Case 1b, $y < d$, and one of two things happens. Either the firm orders $d - y$ if $d - y \leq R$, or else it orders R. In the first case, the new IL_{2s} is zero. In the second case, it is $-(d - y - R)$.

Combining Case 1a and 1b, we have:

$$E[IL_{2s} \mid success] = \int_{d}^{\infty} (y-d) f_y(y) dy - \int_{-\infty}^{d-R} (d-y-R) f_y(y) dy \tag{28}$$

A.3.2 Case 2: Failed Delivery Period

In the $(j-1)^{st}$ disrupted period ($j \geq 1$), $IL > 0$ iff the initial supply from the primary supplier exceeded j periods of demand. This is Case 2a. Thus for $j \geq 1$ we have:

$$E[IL_{2s}^{+} \mid 2a] = \int_{jd}^{\infty} (y-jd) f_y(y) dy \tag{29}$$

Similarly, if $y \in (0, d-R)$, in Case 2b, then the number of backorders (IL^{-}) in the $(j-1)^{st}$ disrupted period is the original deficit $d-R-y$, plus an additional deficit of $d-R$ for each of the $j-1$ disrupted periods. Thus for any $y \in (0, d-R)$, $IL_{2s}^{-} = j(d-R) - y$, and we have:

$$E[IL_{2s}^{-} \mid 2b] = \int_{-\infty}^{d-R} (j(d-R)-y) f_y(y) dy \tag{30}$$

Suppose instead that the initial delivery provided enough stock to last for k periods, but that after k failures (the $(k+1)^{st}$ period after the delivery), the firm cannot make up the difference with R; this is Case 2c, $y \epsilon (kd, (k+1)d-R)$. Then after $j-1$ failures, $j > k$, the inventory deficit $(jd-(j-k)R)-y$. Thus for $k \leq j-1$ we have:

$$E[IL_{2s}^{-} \mid 2c] = \int_{kd}^{(k+1)d-R} (jd-(j-k)R-y) f_y(y) dy \tag{31}$$

Finally, consider Case 2d: $y \in (kd-R, kd)$. Then the firm ends the $(k-1)^{th}$ disrupted period with $IL_{2s} = 0$ (since the reserve can be used to make up the shortfall), while it ends the $(j-1)^{st}$ disrupted period, $j > k$, with $IL_{2s} = (j-k)(d-R)$. Thus for $k \leq j-1$ we can write this as:

$$E[IL_{2s}^{-} \mid 2d] = \int_{kd-R}^{kd} (j-k)(d-R) f_y(y) dy \tag{32}$$

A.3.3 Inventory Level Expectation

Only the first integral from (28) and equation (29) yield a positive inventory level; therefore,

$$E[IL_{2s}^{+}] = \sum_{j=1}^{\infty} \left[\pi_{j-1} \int_{jd}^{\infty} (y-jd) f_y(y) dy \right] \tag{33}$$

We combine the remaining integrals to get:

$$E[IL^-_{2s}] = \sum_{j=1}^{\infty}\left[\pi_{j-1}\int_{-\infty}^{d-R}(j(d-R)-y)f_y(y)dy\right]+ \tag{34}$$

$$(1-\alpha)\sum_{j=2}^{\infty}\sum_{k=1}^{j-1}\left[\alpha^{j-1}\left(\int_{kd-R}^{kd}(j-k)(d-R)f_y(y)dy\ +\right.\right. \tag{35}$$

$$\left.\left.\int_{kd}^{(k+1)d-R}(jd-(j-k)R-y)f_y(y)dy\right)\right] \tag{36}$$

□

A.4 Proof of Proposition 5, Section 4.2.3

We will first reduce $E[IL_{2s}]$ and then $C_{2s}(s,R)$.

A.4.1 Reduction of $E[IL_{2s}]$

Applying Lemma 8, we have:

$$\begin{aligned}E[IL^+_{2s}] &= \sum_{j=1}^{\infty}\left[\pi_{j-1}\int_{jd}^{\infty}(y-jd)f_y(y)dy\right]\\ &= \sum_{j=1}^{\infty}\left[\pi_{j-1}\int_{jd-s}^{\infty}(w-(jd-s))f(w)dw\right]\\ &= \sum_{j=1}^{\infty}\left[\pi_{j-1}G(jd-s)\right]\end{aligned} \tag{37}$$

Now we examine each integral from $E[IL^-_{2s}]$ individually or in small groups. Line (34) reduces as follows:

$$\sum_{j=1}^{\infty}\left[\pi_{j-1}\int_{-\infty}^{d-R}(j(d-R)-y)f_y(y)dy\right]=$$

$$\sum_{j=1}^{\infty}\left[\pi_{j-1}\int_{-\infty}^{d-R-s}((j-1)(d-R)+d-R-s-w)f(w)dw\right]=$$

$$\sum_{j=1}^{\infty}\left[\pi_{j-1}(j-1)(d-R)F(d-R-s)-\overline{w}+d-R-s+G(d-R-s)\right]=$$

$$d-R-s-\overline{w}+G(d-R-s)+(d-R)F(d-R-s)\sum_{j=1}^{\infty}\left[\pi_{j-1}(j-1)\right] \tag{38}$$

For the integral from line (35) we have:

$$\int_{kd-R}^{kd}(j-k)(d-R)f_y(y)dy=\int_{kd-R-s}^{kd-s}(j-k)(d-R)f(w)dw=$$

$$(j-k)(d-R)\left(F(kd-s)-F(kd-R-s)\right) \tag{39}$$

Rewrite the integral from line (36) as:

$$\int_{kd}^{(k+1)d-R}(jd-(j-k)R-y)f_y(y)dy =$$

$$\int_{kd-s}^{(k+1)d-R-s}(jd-(j-k)R-s)f(w)dw - \int_{kd-s}^{(k+1)d-R-s}wf(w)dw \tag{40}$$

The first integral in the right-hand-side of (40) is simply:

$$(jd-(j-k)R-s)\left(F((k+1)d-R-s)-F(kd-s)\right) \tag{41}$$

Applying (23) from Lemma 8, the second integral in (40) is equal to:

$$\overline{w}-((k+1)d-R-s)(1-F((k+1)d-R-s))-G((k+1)d-R-s)-$$

$$(\overline{w}-(kd-s)(1-F(kd-s))-G(kd-s)) =$$

$$-(d-R)+((k+1)d-R-s)F((k+1)d-R-s)-(kd-s)F(kd-s)-G((k+1)d-R-s)+G(kd-s) \tag{42}$$

Combining (41) and (42) gives the following for the integral from line (36):

$$(j-k-1)(d-R)F((k+1)d-R-s)-(j-k)(d-R)F(kd-s)+$$

$$(d-R)+G((k+1)d-R-s)-G(kd-s) \tag{43}$$

Combining (39) and (43),

$$\begin{aligned}(39)+(43) &= (j-k)(d-R)\left(F(kd-s)-F(kd-R-s)\right)+\\ &\quad (j-k-1)(d-R)F((k+1)d-R-s)-(j-k)(d-R)F(kd-s)+\\ &\quad (d-R)+G((k+1)d-R-s)-G(kd-s)\\ &= (d-R)(1-(j-k)F(kd-R-s)+(j-k-1)F((k+1)d-R-s))+\\ &\quad G((k+1)d-R-s)-G(kd-s)\end{aligned} \tag{44}$$

Substituting (38) and (44), into (34)-(36), we have:

$$\begin{aligned}E[IL_{2s}^{-}] &= d-R-s-\overline{w}+G(d-R-s)+(d-R)F(d-R-s)\sum_{j=1}^{\infty}[\pi_{j-1}(j-1)]+\\ &\quad \sum_{j=2}^{\infty}\sum_{k=1}^{j-1}\Big[\pi_{j-1}\Big(G((k+1)d-R-s)-G(kd-s)+\\ &\quad (d-R)(1-(j-k)F(kd-R-s)+(j-k-1)F((k+1)d-R-s))\Big)\Big]\end{aligned} \tag{45}$$

A.4.2 Reduced form for $C_{2s}(s,R)$

Rewriting (14) in terms of w, we have:

$$\begin{aligned} C_{2s}(s,R) &= p_2 \sum_{i=1}^{\infty} \pi_{i-1} \left(\int_{-\infty}^{id-R-s} Rf(w)dw + \int_{id-R-s}^{id-s} (id-s-w)f(w)dw \right) + \\ & C_o E[IL_{2s}]^+ + C_u E[IL_{2s}]^- + \pi_0 p_1 \left(s - E[IL_{2s}]\right) + rR \end{aligned} \tag{46}$$

Applying Lemma 8, the two integrals in (46) can be written as:

$$\begin{aligned} & \int_{-\infty}^{id-s} (id-s-w)f(w)dw - \int_{-\infty}^{id-R-s} (id-R-s-w)f(w)dw = \\ & -\overline{w} + id - s + G(id-s) + \overline{w} - (id-R-s) - G(id-R-s) = \\ & R + G(id-s) - G(id-R-s) \end{aligned} \tag{47}$$

Combining (46) and (47) yields the expression in (15).

□

A.5 Proof of Proposition 7, Section 4.3

To solve for the optimal decision variables, we set the first derivatives of (18) equal to zero. Applying (22) from Lemma 8, we have:

$$\begin{aligned} \frac{\partial}{\partial R} C_{2s,t}(s,R) &= r + p_2 - C_u + (1-\alpha)\left[-C_u \left(F\left(d-R-s\right)-1\right) + \right. \\ & \left. p_2 \left(F\left(d-R-s\right)-1\right)\right] \\ &= r - \alpha(C_u - p_2) - (1-\alpha)(C_u - p_2)F\left(d-R-s\right) \\ &= 0 \end{aligned} \tag{48}$$

$$\Leftrightarrow d - R - s = F^{-1}\left(\frac{r-\alpha(C_u-p_2)}{(1-\alpha)(C_u-p_2)}\right) \tag{49}$$

$$\Leftrightarrow R_t = d - s - F^{-1}\left(\frac{r-\alpha(C_u-p_2)}{(1-\alpha)(C_u-p_2)}\right) \tag{50}$$

In addition,

$$\begin{aligned} \frac{\partial}{\partial s} C_{2s,t}(s,R) &= (1-\alpha)\left[-C_u F\left(d-R-s\right) + p_2\left(F\left(d-R-s\right) - F\left(d-s\right)\right) \right. \\ & \left. -C_o F\left(d-s\right) + C_o + p_1\right] \end{aligned} \tag{51}$$

Since the reservation quantity must be non-negative, R_t is the maximum of the right-hand-side of (50) and zero. Suppose first that $R_t > 0$. Rearranging (50) and taking $F(\cdot)$ of both sides, we get

$$F\left(d - R_t^* - s\right) = \frac{r-\alpha(C_u-p_2)}{(1-\alpha)(C_u-p_2)} \tag{52}$$

Substituting (52) into (51), we get

$$\begin{aligned}\frac{\partial}{\partial s}C_{2s,t}(s,R) &= (1-\alpha)\left[-C_u\left(\frac{r-\alpha(C_u-p_2)}{(1-\alpha)(C_u-p_2)}\right)+\right.\\ &\left. p_2\left(\left(\frac{r-\alpha(C_u-p_2)}{(1-\alpha)(C_u-p_2)}\right)-F\left(d-s\right)\right)-C_oF\left(d-s\right)+C_o+p_1\right]\\ &= 0 \qquad (53)\end{aligned}$$

$$\Leftrightarrow -(C_u-p_2)\left(\frac{r-\alpha(C_u-p_2)}{(1-\alpha)(C_u-p_2)}\right)+C_o+p_1=(C_o+p_2)F\left(d-s\right) \qquad (54)$$

$$\Leftrightarrow s_t^*=d-F^{-1}\left(\frac{\alpha(C_u-p_2)-r+(1-\alpha)(C_o+p_1)}{(1-\alpha)(C_o+p_2)}\right) \qquad (55)$$

This allows us to write R_t^* independent of s_t^*:

$$R_t^*=max\left\{0,F^{-1}\left(\frac{\alpha(C_u-p_2)-r+(1-\alpha)(C_o+p_1)}{(1-\alpha)(C_o+p_2)}\right)-F^{-1}\left(\frac{r-\alpha(C_u-p_2)}{(1-\alpha)(C_u-p_2)}\right)\right\} \qquad (56)$$

Now if $R_t=0$, we return to equation (51):

$$\begin{aligned}\frac{\partial}{\partial s}C_{2s,t}(s,R) &= (1-\alpha)\left[-C_uF\left(d-s\right)+p_2\left(F\left(d-s\right)-F\left(d-s\right)\right)\right.\\ &\left.-C_oF\left(d-s\right)+C_o+p_1\right]\\ &= (1-\alpha)\left[C_o+p_1-(C_o+C_u)F\left(d-s\right)\right]\\ &= 0 \qquad (57)\end{aligned}$$

$$\Leftrightarrow F\left(d-s\right)=\frac{C_o+p_1}{C_o+C_u} \qquad (58)$$

$$\Leftrightarrow s_t^*=d-F^{-1}\left(\frac{C_o+p_1}{C_o+C_u}\right) \qquad (59)$$

Thus we can combine (55) and (59) to write s_t^* as the following:

$$s_t^*=\begin{cases} d-F^{-1}\left(\frac{\alpha(C_u-p_2)-r+(1-\alpha)(C_o+p_1)}{(1-\alpha)(C_o+p_2)}\right), & \text{if } R_t^*>0;\\ d-F^{-1}\left(\frac{C_o+p_1}{C_o+C_u}\right), & \text{if } R_t^*=0.\end{cases} \qquad (60)$$

□