Models for Production Planning under Power Interruptions

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Abstract

We present a robust optimization model for production planning under electricity supply that is subject to uncertain interruptions caused by participation in interruptible load contracts (ILCs). The objective is to minimize the cost of electricity used for production while providing a robust production plan which ensures demand satisfaction under all possible interruption scenarios. Due to the combinatorial size of the interruption scenario set, this is a challenging problem. Furthermore, we assume that no probabilistic information is known about the supply uncertainty and we only use the information given in the ILC to structure an uncertainty set that captures the possible scenarios. We first construct a general robust framework to handle this uncertainty and provide a heuristic algorithm which approximates the optimal solution of the robust model. Computational experiments are also conducted to compare the performance of the robust model and the heuristic algorithm. Next we show how to include operational impacts of interruptions such as "recovery modes" in the definition of the uncertainty set.

1 Introduction

Under deregulation, the electricity industry is continuously evolving and changing as different markets, such as the derivative, forward and spot markets, become more common. Within these markets, the availability of diversified services and pricing menus is increasing. Participants in these markets have developed and adopted

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many financial instruments for electricity transactions so that electricity is produced, priced and traded more efficiently. Interruptible load contracts (ILCs) are one type of these instruments which are employed to increase demand-side involvement so that the adverse effects of supply shortages at the utility are mitigated.

As defined in [2], an interruptible load contract between a utility and an industrial/commercial customer allows the utility to interrupt part or all of the supply of electricity to the customer over some period of time in exchange for some form of monetary reward. Typically, the maximum number of interruptions that can occur during the specified time interval is defined in the contract. Usually the utility does not physically interrupt the customer, but rather gives the customer advance notice to reduce loads or face a significantly increased cost rate during the interruption period.

Companies that participate in such ILCs benefit from the discounted rate structure, while the electricity utility enjoys the benefits of priority service. As defined in [15], priority service refers to a menu of contingent contracts in which service provision is prioritized according to the customer's valuation of the service under supply scarcity. This helps the utility to reduce its exposure to spot market prices in the event of supply shortages. However, customers now bear additional risks that arise from the interruption of power, such as backorder costs, reduced production and storage capacities. Participation in ILCs requires flexibility from customers in order to be able to honor this requirement along with their requirement to satisfy the demand. To do this, customers must account for potential interruptions in their production plans. As demonstrated in a report by [12], some companies that participated in ILCs were unable to honor the load reduction requirement due to a lack of capabilities and/or a lack of necessary planning. This was one of the triggering events that caused the California electricity crisis in 2001 [12].

The purpose of this paper is to provide a production planning framework for rate-paying industrial production companies whose production operations strongly depend on electricity. The problem we study is an operational-level, aggregate production and inventory planning problem with electricity supply uncertainty and deterministic demand. In particular, our methodology provides a feasible production plan, if one exists under the given production and storage capabilities, that satisfies customer demands under all possible interruption scenarios. Furthermore, we avoid using any probability distribution to characterize the interruptions, since our objective is to guarantee feasibility in any possible interruption realization, and since the distribution of interruptions is typically unknown. Therefore we use the only pieces of information that are available to the industrial company through the ILC: the length of the planning horizon (which is equal to the duration of the ILC), the unit length of interruptions, the limit on the number of interruptions and the reward scheme.

Our modeling approach has several benefits. First, the interruption uncertainty framework we describe allows different contract rules and operational rules to be embedded in the production planning problem simultaneously. Companies might have different operational procedures in the event of interruptions, such as limiting the production in post-interruption recovery or prohibiting production level increases in some periods. Second, the methodology we describe can be used for different types of ILCs, possibly with different reward schemes, such as the pay-in-advance and pay-as-you-go schemes described by [2]. Third, information regarding the utility's interruption dispatch behavior can easily be embedded into our production planning framework.

Although a number of papers on ILCs have appeared in the literature (see §2.1), these papers all consider the problem faced by the interrupting party, that is, the power provider which can be a power generator or the utility. To the best of our knowledge, ours is the first paper to consider the impact of ILCs on the production planning process of the interrupted party, that is, the industrial company.

Our study is motivated by an air-separation process, and many of the elements in our model come from that setting; however, the robust methodology we provide is not restricted to this setting. In an air-separation process, a mass production system is used to produce liquid nitrogen and liquid oxygen in large volumes using special-purpose equipment. Both products are produced simultaneously through a single process. Virtually the only raw material needed is air, which is available in very large quantities at a very low cost, leaving electricity as the most critical resource that is required for production. The products generated by the air separation process are critical for medical treatment in hospitals, as well as for other high-impact applications. The availability of the products is therefore, literally, a life-and-death situation, and this motivates the assumption in our model that stockouts are not allowed under any interruption scenario.

The remainder of this paper is organized as follows. We review the literature on ILCs and production planning under supply uncertainty in §2. In §3, we formulate the robust production planning problem under supply uncertainty. §4 gives a real world application of the robust model in which the *uncertainty set* that is constructed to include interruption-related operational rules. Computational results for both models are discussed in §5. §6 concludes the paper and discusses future research.

2 Literature Review

2.1 Interruptible Load Contracts

The literature on ILCs is focused on pricing and has its roots in the body of research on service priority [15, 37]. The former paper studies the impact of the division of customers into priority classes in which interruptible service is priced according to service reliability. The latter extends this study by adding an optimal price menu for better segmentation. [36] further extends these studies by introducing an early notification option. [13] addresses the optimal interruption dispatch behavior of the utility within a stochastic dynamic programming (SDP) framework when there is a limit on the interruptions that can be called.

[24] studies the design and pricing of financial contracts for interruptible electricity service, emulating three types of ILCs using financial instruments such as forwards and options. [2] further extends the works by [13, 24] and creates a structural model that is calibrated using temperature data. This model is then embedded into an SDP framework, which is used to valuate the ILC and to find the optimal interruption dispatch policy under different reward schemes. The authors also explore the performance of ILCs under retailer competition. [19, 20] study incentive-compatible ILCs and describe a methodology for the electricity utility to estimate the customer demand and the value of interruptibility for the customer through utility data. The authors' objective is to design the contract incentives in a way that enables the revelation of the customers' actual valuation of the interruption.

All of the above works address ILCs from the contract-issuing party's perspective. To the best of our knowledge, there have been no studies that explore the impact of participation in an ILC on production planning from the contract-taking industrial company's perspective.

2.2 Production Planning Literature

According to [32], decision-making can be classified into three different categories: certainty, risk, and uncertainty. In the certainty case, all decision-making elements are deterministic and known. Risk and uncertainty arise when the complete information that describes a situation to its full extent is lost. In risk situations, the information that the decision-maker has still contains some descriptive elements, such as probability distributions, that explain the situation to some extent. In the absence of these descriptive elements, uncertainty arises. Consistent with these definitions, we use the term "uncertainty" throughout this paper to refer to problems in which no probabilistic information is available about the random parameters. Our problem is an operational-level, aggregate production and inventory planning problem with supply uncertainty.

Production planning under supply uncertainty has received a lot of attention from the supply chain, production and inventory theory communities. For example, [14] develops an optimal dynamic inventory policy in the presence of market disruptions. The model is based on the framework of a continuous-time Markov decision process with a finite state space in which the rate of inventory accumulation or reduction can be continuously adjusted. An economic analysis is provided for both elastic and inelastic demand. [38] develops an optimal inventory policy for EOQ systems where the start time of the disruption is known beforehand but the duration is unknown. [30] studies a continuous-review stochastic inventory problem with random demand and random lead-time where supply availability is an alternating renewal process; see also [22, 28, 29]. [1] considers a continuous-review inventory system in which partial backorders are allowed; see also [26]. [21] studies a periodic-review inventory model, shows the optimality of an order-up-to policy, and obtains a newsboy-like formula that determines the optimal order-up-to levels under deterministic dynamic demand and stochastic supply unavailability.

Most of the related work on supply uncertainty in the supply chain literature focuses on either random yields in the supply processes or complete supply disruptions, with known probabilistic information about disruptions. We refer the reader to [31, 34, 39] for through reviews of the literature on supply uncertainty under various supply chain settings. For a general review of production planning under uncertainty, see [27]. To the best of our knowledge, production planning under anticipation of the supply uncertainty, with no probabilistic information, has never been explicitly addressed.

2.3 Robust Optimization

One of the seminal works in robust optimization is by Soyster [35], who proposed a linear model to construct a solution that is feasible for all data that belong to a convex set. Optimizing over the worst-case scenario might produce solutions that are too conservative, and Soyster's approach has been criticized for being ultraconservative by [10]. To cure over-conservativeness, [3, 4, 17, 18] consider uncertain convex optimization problems under ellipsoidal uncertainty sets. One drawback of this methodology is that the complexity of the original problem, with no uncertainty included, is increased when it is transformed into its robust counterpart (RC). To a limited extent, [25] extend robust optimization for inclusion of discrete variables. When the objective is to optimize the worst-case performance over a set of scenarios, they show that the RCs of many polynomially solvable discrete optimization problems are NP-hard. Methodologies other than Soyster's require us to make additional assumptions on the structure of the uncertainty set and in this study we focus on uncertainty sets that can be directly constructed from contractual rules. Therefore, we construct an uncertainty polyhedron using only the contract rules, and for this type of construction Soyster's methodology is appropriate. Another benefit of using this approach is the ease of incorporating the impact of interruptions on production modes.

[9–11] introduce and study the "budget of uncertainty" concept. This approach provides a mechanism to control the level of conservativeness by allowing only a subset (the size of which is controlled by the "budget of uncertainty") of the uncertain parameters to deviate from their nominal values simultaneously. Furthermore, the RC preserves the complexity of its nominal problem and thus can easily be extended to discrete optimization problems.

The studies we have mentioned until now are all static decision-making problems, in the sense that all the decision variables are determined before the realization of uncertainty, i.e. "here and now" decisions. [6] introduce and study the adjustable RC (ARC) of multistage uncertain linear programming (LP) problems in which some of the decisions can be delayed until after some of the uncertain parameters have been observed, i.e. "wait-and-see decisions." The authors use the intersection of ellipsoids to define the uncertainty set and find that, often, the ARC is significantly less conservative than the usual RC. However, in most cases the ARC is NP-hard. To address this issue, the authors introduce affinely adjustable RCs (AARC) in which the wait-and-see decisions are formulated as affine functions of the uncertain parameters. [7] further extends AARCs to include controlled deterioration in performance for large deviations in the uncertain data. For detailed theoretical background we refer to [5, 8].

3 Robust Production-Planning Models

3.1 Electricity Supply, Prices and Interruptions

Following the setting in [2], we allow the utility to place multiple interruptions over the planning horizon. We assume that every interruption lasts for exactly one period; however, multiple consecutive interruptions are allowed, as long as the total number of interrupted periods does not exceed the maximum specified by the contract. The uncertainty in the model stems from a lack of information regarding the exact timing of interruptions. Furthermore, we assume there is no prior historical information available to the company regarding interruptions. Moreover, we assume that the production company has production equipment that can be shut down instantly; therefore, there are no depreciation- and maintenance-related costs caused by interruptions. Other than the machinery and labor, the majority of the variable production cost is comprised by electricity consumption. Typically, air separation plants, aluminum foundries and paper mills have such cost characteristics [33]. We also assume that the company can only purchase electricity from the utility and that electricity is only used for production.

3.2 Production Setting

We consider a company that has multiple plants and, due to technological and physical reasons, production and storage capacities of the plants are limited. All processing tasks are performed in batch mode and the processing time is negligible; however, there is a finite daily production capacity. The plants are geographically distinct, but each plant has the capability to serve the other plants' customers. This means the demand for products is aggregated over all plants. This is depicted in Figure 1 for a two-plant example.

In general, the utility charges a fixed rate for electricity, as is typical of a fixed-price retail contract. As an incentive for participating in the ILC, the electricity retailer offers two types of rewards: pay-as-you-go and pay-in-advance, as defined in [2]. In a pay-as-you-go scheme, the utility pays a penalty for each interruption it dispatches. In a pay-in-advance scheme, there is no per-interruption penalty for the utility, however, the utility provides an overall discount in the electricity price for the contract horizon. For both payment schemes, the maximum number of interruptions that can be dispatched is contractually defined, though the exact times of the interruptions are not known in advance (by either party). In this study, we assume that the industrial

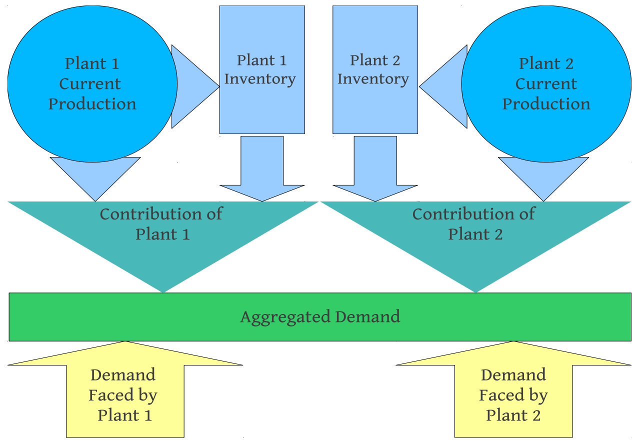

Figure 1: Demand Aggregation and Contribution of Plants

company has accepted the pay-in-advance scheme, and therefore it faces a discounted fixed rate, r.

In our setting, the industrial company has negotiated a joint ILC contract for all plants, and this contract stipulates that at most one plant will be interrupted in a given period, where interruptions last exactly one period. The production planning horizon is equal to the contract horizon, which is terminated by the expiration of the contract. The objective of the industrial company is to minimize production costs subject to the following constraints:

- Daily production in each plant is less than or equal to the daily capacity;
- Inventory capacity is limited;
- Daily aggregated demand must be satisfied through production and accumulated inventory, i.e., no stock-outs are allowed.

The industrial company plans its production anticipating the interruptions, and the production plan is set at the beginning of the planning horizon. Once the production plan is set, it cannot be changed until the end of the horizon. In particular, the production plan cannot be changed in reaction to interruptions, other than to zero out the production at an interrupted plant. A robust solution for this problem has one important characteristic: It stays feasible under all possible interruption scenarios. We use these characteristics as our foundation for the robust modeling approach. A production planner can choose to “robustify” the aggregated inventory over plants, i.e., make sure that the sum of the inventory levels at different plants stays non-negative throughout the horizon, or s/he can choose to “robustify” individual inventories, i.e., make sure that individual inventories at plants are never exhausted throughout the horizon. In our model, we robustify the aggregated inventory over plants, but we describe how the same logic can be used to robustify individual inventories in §3.4. Robustifying individual inventories produces more conservative and potentially more costly solutions given that inventory at the other plants can no longer be used as a buffer for the adverse effects of an interruption at a particular plant. We enforce the at-most-one-interruption-per-period rule and denote the maximum number of interruptions as K. We use the notation in Table 1 throughout the remainder of the paper.

Indices & Sets	
p	plant index, $p \in \mathcal{P} := \{1, \ldots, P\}$
t	time index, $t \in \mathcal{T} := \{1, \ldots, T\}$
g	product index, $g \in \mathcal{G} := \{1, \ldots, G\}$
$\mathcal{U}$	uncertainty set
Parameters	
$c^{pro}_{p,g}$	production capacity at plant p for product g
$c^{inv}_{p,g}$	inventory capacity at plant p for product g
$inv_{0,p,g}$	initial inventory of product g at plant p
$d_{t,g}$	aggregated demand for product g in period t
K	maximum number of interruptions
v	power-to-unit conversion factor (units/kWh)
r	electricity price ($/kWh)
Variables	
$z(\cdot)$	objective function value
$x_{t,p,g}$	amount of product g produced at plant p in period t
$w_{t,p,g}$	amount of demand for product g satisfied by plant p in period t
$inv_{t,p,g}$	inventory of product g at plant p at the end of period t[1]

Table 1: Notation

3.3 Deterministic Production Planning Model

In this section, we present the deterministic production planning model, which contains no interruptions. We will use this model as a baseline and introduce the impact of interruptions in §3.4.

Note that Table 1 contains two sets of production-related decision variables, $x_{t,p,g}$ and $w_{t,p,g}$. The former represents the production of product g at plant p in period t, while the latter represents the demand for product g in period t that is satisfied by plant p. The two quantities may differ because production in period t may be used to satisfy demand *or* to be stored in inventory for future periods. The w variables are represented in Figure 1 by the arrows labeled "Contribution of Plant p."

Using this notation, we characterize the inventory levels that accumulate in each time period in (1a)–(1b). In (1a), the ending inventory in period t is calculated from the initial horizon inventory by adding the total production and subtracting the total items used to satisfy demand through period t. (Recall that the plants maintain separate inventories.) Equation (1b) reflects the relationship between the ending inventories in periods $t-1$ and t.

$$inv_{t,p,g} = \sum_{i=1}^{t} x_{i,p,g} - \sum_{i=1}^{t} w_{i,p,g} + inv_{0,p,g} \tag{1a}$$

$$= inv_{t-1,p,g} + x_{t,p,g} - w_{t,p,g} \quad \forall t,p,g \tag{1b}$$

We first present the deterministic production planning model, which we refer to as the Outer Problem (OP). (An "inner problem", solved by the interrupting party, will be discussed in §3.4.)

$$OP\colon\ \min \sum_{p=1}^{P}\sum_{t=1}^{T}\sum_{g=1}^{G} vrx_{t,p,g} \tag{2a}$$

$$\text{s.t. } inv_{t,p,g} = inv_{t-1,p,g} + x_{t,p,g} - w_{t,p,g} \quad \forall t,p,g \tag{2b}$$

$$\sum_{p=1}^{P} w_{t,p,g} \geq d_{t,g} \quad \forall t,g \tag{2c}$$

$$x_{t,p,g} \leq c_{p,g}^{pro} \quad \forall t,p,g \tag{2d}$$

$$\sum_{p=1}^{P} inv_{t,p,g} \geq 0 \quad \forall t,g \tag{2e}$$

$$inv_{t,p,g} \leq c_{p,g}^{inv} \quad \forall t,p,g \tag{2f}$$

$$x_{t,p,g}, w_{t,p,g} \geq 0 \quad \forall t,p,g \tag{2g}$$

The objective function (2a) is simply the cost of electricity used for production—the multiplier v converts production units to electricity consumption (in kWh). Constraints (2d) and (2f) enforce the production and inventory capacities. Constraints (2b) and (2c) enforce the relationship between actual demand, $d_{t,p,g}$, and the demand-satisfaction variable, $w_{t,p,g}$. In particular, for each time period and product, the sum of the units coming from all plants must equal the aggregated demand. Constraints (2e) enforce the no-stock-out condition on pooled inventory. Finally, constraints (2g) are non-negativity constraints.

3.4 Robust Simple Model

In this section, we robustify the OP against interruptions, giving rise to a problem we refer to as the Robust Outer Problem (ROP). Since we have no information regarding the utility's interruption dispatch policy, the robust problem must ensure feasibility in all possible interruption scenarios. An interruption scenario is defined by the times and locations of all K interruptions and may be constructed as follows. One would first choose K interruption times from the set $t = 1, ..., T$. There are $\binom{T}{K}$ such choices of interruption times. For a given interrupted time period, exactly one of the plants must be interrupted (due to the contractual agreement). There are P^K such possibilities. Therefore, there are exactly $\binom{T}{K}P^K$ interruption scenarios, which is combinatorial in size. The stochastic programming approach, which depends on individual scenarios rather than on a description of the uncertainty set, is computationally expensive. Hence, we use the robust optimization approach for handling the uncertainty. Note that the analysis above assumes that a scenario has exactly K interruptions. It is possible that fewer than K interruptions will occur over the horizon, but since we are interested in optimizing over the worst case, it is sufficient to assume that exactly K interruptions occur.

The uncertainty set, $\mathcal{U}$, contains all of the uncertainty scenarios. Our objective is to ensure that the aggregate inventory level is non-negative in every scenario. Another way to think about $\mathcal{U}$ is as the feasible set of an optimization problem aimed at determining the minimum inventory levels as a function of the (a priori unknown) interruptions and the production levels given by x. Consequently, we introduce a separate class of variables, $\xi_{t,p}$, to model the utility's choice of interruptions, and we describe the aggregate inventory as function of those variables subject to constraints derived from the contract clauses. The $\xi_{t,p}$ are defined as

$$\xi_{t,p} = \begin{cases} 1 & \text{if an interruption occurs in period } t \text{ at plant } p \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\mathcal{U} = \left\{ \xi \in \{0,1\}^{T \times P} \middle| \sum_{p=1}^{P} \xi_{t,p} \leq 1 \;\; \forall t, \;\; \sum_{p=1}^{P} \sum_{t=1}^{T} \xi_{t,p} \leq K \right\}.$$

Each $\xi \in \mathcal{U}$ corresponds to an interruption scenario since the vector ξ characterizes the time and location of every interruption. The first constraint ensures that no simultaneous interruptions occur, while the second constraint ensures that the number of interruptions doesn't exceed the contractual limit. Therefore, we replace the inventory constraint (2e) with

$$\min_{\xi \in \mathcal{U}} \left\{ \sum_{p=1}^{P} inv_{t,p,g}(\xi) \right\} \geq 0 \ \ \forall t, g \tag{4}$$

This approach allow us to embed the uncertainty into the OP to obtain the ROP, which is formulated below in (5a)–(5g).

This relationship between the company and the utility can be thought of as a leader-follower game in which the utility is the follower. This is reflected in the relationship between (4) and the ROP, formulated in (5a)–(5g) below. In particular, the company solves the ROP at the beginning of the planning horizon. This problem anticipates the interruptions and determines optimal production levels while maintaining feasibility in all interruption scenarios. The scenarios are determined by a virtual "opponent" whose objective is to sabotage the company's production so that the demand is not satisfied. The opponent does this by solving an inner optimization problem (IOP), which aims to cause infeasibilities by placing interruptions carefully throughout the time horizon. The IOP for a given t and g is given by (4). This problem is embedded into the constraints of the ROP; thus, the ROP contains constraints that can only be instantiated, evaluated, and enforced by solving the optimization problems described in (4). Note that the feasible region that contains the opponent's possible interruption decisions is not affected by the production decisions made by the company in the ROP.

Once a plant is interrupted, all production ceases. The ROP anticipates the opponent's behavior, so that all "optimal" actions of the opponent (worst-case scenarios for the planner) are considered and a production plan that ensures non-negative inventory, which implies feasibility in all possible scenarios, is found if one exists. This is characterized in constraint (5e) of the ROP. Furthermore, all outer problem variables are regarded as parameters in all inner problems.

The opponent solves as many optimization problems as there are inventory pools; that is, one for each $(t, g) \in \mathcal{T} \times \mathcal{G}$. The feasible region is the same for all IOPs; it is constructed using only the contractual obligations on interruptions. However the objective functions of a given IOP is the inventory level of the corresponding product at the corresponding time period, which is jointly characterized by production and interruption decisions. A negative inventory level for some (t, g) would mean success for the opponent since it would imply a stock-out for the company, which in turn implies the infeasibility of the production plan. The set of feasible actions of the opponent is characterized by the uncertainty set, $\mathcal{U}$, and each scenario in this uncertainty set corresponds to a possible action in the arsenal of the opponent.

Since production and inventory decisions belong to the planner while interruption decisions belong to the electricity utility, the bi-level modeling approach is appropriate for our problem. We formulate the bi-level form robust outer problem (ROP) as:

$$ROP\colon \ \min \sum_{p=1}^{P}\sum_{t=1}^{T}\sum_{g=1}^{G} vrx_{t,p,g} \tag{5a}$$

$$\text{s.t.} \ \ w_{t,p,g} \le inv_{t-1,p,g} + x_{t,p,g} \ \ \forall t,p,g \tag{5b}$$

$$\sum_{p=1}^{P} w_{t,p,g} \ge \sum_{p=1}^{P} d_{t,p,g} \ \ \forall t,g \tag{5c}$$

$$x_{t,p,g} \le c^{pro}_{t,p} \ \ \forall t,p,g \tag{5d}$$

$$\min_{\xi\in\mathcal{U}} \left\{ \sum_{p=1}^{P} inv_{t,p,g}(\xi) \right\} \ge 0 \ \ \forall t,g \tag{5e}$$

$$inv_{t,p,g} \le c^{inv}_{p,g} \ \ \forall t \tag{5f}$$

$$x_{t,p,g}, w_{t,p,g} \ge 0 \ \ \forall t,p,g \tag{5g}$$

Constraints (5e) state that, for each t and g, the pooled inventory at plants, $\sum_{p=1}^{P} inv_{t,p,g}$, must be non-negative in every possible interruption scenario ξ. To robustify the individual inventories instead of the pooled inventory, one can replace constraints (5e) with

$$\min_{\xi\in\mathcal{U}} \{inv_{t,p,g}(\xi)\} \ge 0 \ \ \forall t,p,g.$$

Other than (5e), the ROP is identical to the OP. Moreover, note that setting $K = 0$ reduces the ROP to the OP.

In general bi-level programs (BLPs) are non-convex [16] due to the fact that feasible regions of lower level problems are not necessarily convex and connected. In our case the ROP is a bi-level mixed-integer problem which is obviously non-convex. However we can represent inventory levels as bilinear functions of $\mathbf{x}$ and ξ

(as in (6,7)), we can convert the BLP into an LP using the optimality conditions of the inner problems; this approach is described in detail later in this section.

The opponent evaluates the quality of his interruption decisions by their effect on the actual production and inventory; however, the actual production is decided by the planner, not the opponent. Therefore, the impact of the opponent's interruption decision on the production plan needs to be represented in the inner problem. To this end, we define an auxiliary variable $\bar{x}_{t,p,g}$ for the inner problem as:

$$\bar{x}_{t,p,g} \quad = (1-\xi_{t,p})x_{t,p,g} \ \ \forall t,p,g \tag{6}$$

Intuitively, $\bar{x}_{t,p,g}$ represents the actual production of product g at plant p in period t—as planned, if there is no interruption ($\xi_{t,p} = 0$) or zero, if there is ($\xi_{t,p} = 1$). Note that from the IOP's perspective only $\xi_{t,p}$ is a variable and $x_{t,p,g}$ is a parameter. For sake of compactness we write the bilinear term $inv_{t,p,g}(\xi)$ as $\overline{inv}_{t,p,g}$. Inserting (6) into (1a) gives the actual objective for each IOP:

$$\overline{inv}_{t,p,g} \quad = \sum_{i=1}^{t} \bar{x}_{i,p,g} - \sum_{i=1}^{t} w_{i,p,g} + inv_{0,p,g}. \tag{7}$$

Then the $IOP(t,g)$ $\forall t,g$ becomes:

$$\min_{\mathcal{U}} \left\{ \sum_{p=1}^{P} \overline{inv}_{t,p,g} \right\}. \tag{8}$$

These inner optimization problems, which capture the uncertainty, need to be solved simultaneously within the ROP. Note that the feasible region, $\mathcal{U}$, is non-empty and bounded; therefore IOPs are always feasible with finite optimum. However, the binary variables ξ prevent using linear duality directly as suggested by Soyster [35]. Instead, we relax the integrality of the uncertainty set to obtain the relaxed uncertainty set, $\mathcal{U}^R$:

$$\mathcal{U}^R = \quad \left\{ \xi \in [0,1]^{T\times P} \middle| \sum_{p=1}^{P} \xi_{t,p} \leq 1 \ \ \forall t, \ \ \textstyle\sum_{p=1}^{P}\sum_{t=1}^{T} \xi_{t,p} \leq K \right\}. \tag{9}$$

The following holds since $\mathcal{U} \subset \mathcal{U}^R$:

$$\min_{\mathcal{U}^R} \left\{ \sum_{p=1}^{P} \overline{inv}_{t,p,g} \right\} \leq \min_{\mathcal{U}} \left\{ \sum_{p=1}^{P} \overline{inv}_{t,p,g} \right\} \quad \forall t,g. \tag{10}$$

Replacing $\mathcal{U}$ with $\mathcal{U}^R$ in (8) gives us the relaxed inner optimization problem (RIOP). This relaxation allows us to convert the ROP from a BLP to an LP (as we will show below) and effectively reduce the complexity of the problem. In general, due to this relaxation, the real impact of interruptions on inventory levels will be amplified. This is clearly demonstrated in inequality (10). An intuitive way of describing this is as follows: with this linear relaxation, the planner no longer perceives the opponent's interruption strategies, ξ, as $0-1$ decisions but continuous decisions in $[0,1]$. The inventory levels obtained by solving the $RIOP$s will be lower bounds for the inventory levels obtained by solving the IOPs. Tightening the uncertainty set using cuts or providing the tightest linear programming relaxation of IOP mitigates the impact of the relaxation. However, it turns out that for this problem, these modifications are not required; the relaxation is actually equivalent to the original problem, because of the following:

Proposition 3.1. *The constraint matrix of IOP is totally unimodular (TU).*

The proof of Proposition (3.1) is given in A. This property completely cures the side-effects of the relaxation, i.e., all basic feasible solutions of IOPs are integral, which implies that the optimal solutions of $RIOP$ are integral. Moreover, there is no duality gap:

$$\min_{\mathcal{U}^R} \left\{ \sum_{p=1}^{P} \overline{inv}_{t,p,g} \right\} = \min_{\mathcal{U}} \left\{ \sum_{p=1}^{P} \overline{inv}_{t,p,g} \right\} \quad \forall (t,g) \tag{11}$$

Therefore,

$$z^*(DRIOP(t,g)) = z^*(RIOP(t,g)) = z^*(IOP(t,g)) \;\; \forall (t,g). \tag{12}$$

Following Soyster's approach and noting that the problems ROP, IOP and $RIOP$ are all bounded, we replace $\mathcal{U}$ with $\mathcal{U}^R$ in constraints (5e) to obtain $RIOP(t,g)$. The explicit form and the dual variables corresponding to the constraints are given below:

$$
\begin{aligned}
RIOP(t,g)\text{:}\ \min\ & \sum_{p=1}^{P}\left((1-\xi_{t,p})x_{t,p,g}+\sum_{\hat{t}=1}^{t-1}\left((1-\xi_{\hat{t},p})x_{\hat{t},p,g}-w_{\hat{t},p,g}\right)+inv_{0,p,g}\right) & \\
s.t.\ & \sum_{p=1}^{P}\xi_{t,p}\le 1\ \ \forall t & \text{Dual: } \beta_{\hat{t}}^{t,g}\le 0\\
& \sum_{p=1}^{P}\sum_{t=1}^{T}\xi_{t,p}\le K & \gamma^{t,g}\le 0\\
& 0\le \xi_{t,p}\le 1\ \ \forall t,p & \theta_{\hat{t},\hat{p}}^{t,g}\le 0
\end{aligned}
$$

The objective function of $RIOP(t,g)$ can be written as:

$$
\sum_{p=1}^{P}\left(\sum_{\hat{t}=1}^{t}(-\xi_{\hat{t},p}x_{\hat{t},p,g})+\sum_{\hat{t}=1}^{t}x_{\hat{t},p,g}-\sum_{\hat{t}=1}^{t-1}w_{\hat{t},p,g}+inv_{0,p,g}\right), \tag{14}
$$

where

$$
\sum_{p=1}^{P}\left(\sum_{\hat{t}=1}^{t}x_{i,p,g}-\sum_{\hat{t}=1}^{t-1}w_{\hat{t},p,g}+inv_{0,p,g}\right) \tag{15}
$$

is a constant from the perspective of $RIOP(t,g)$. The dual variables (β,γ,θ) are superscripted with t,g since they belong to the problem for given t and g, i.e. one IOP must be solved for each t,g. The explicit form of the dual problem, called $DRIOP(t,g)$, is given below:

$$
\begin{aligned}
DRIOP(t,g)\text{:}\ \max\ & K\gamma^{t,g}+\sum_{\hat{t}=1}^{T}\left(\beta_{\hat{t}}^{t,g}+\sum_{\hat{p}=1}^{P}\theta_{\hat{t},\hat{p}}^{t,g}\right)+ & \\
& \sum_{p=1}^{P}\left(\sum_{\hat{t}=1}^{t}x_{\hat{t},p,g}-\sum_{\hat{t}=1}^{t-1}w_{\hat{t},p,g}+inv_{0,p,g}\right) & \\
s.t.\ & \gamma^{t,g}+\beta_{\hat{t}}^{t,g}+\theta_{\hat{t},\hat{p}}^{t,g}\ \le -x_{\hat{t},\hat{p},g} & \forall\hat{t}\le t,\ \ \forall\hat{p}\\
& \gamma^{t,g}+\beta_{\hat{t}}^{t,g}+\theta_{\hat{t},\hat{p}}^{t,g}\ \le 0 & \forall\hat{t}> t,\ \ \forall\hat{p}\\
& \gamma^{t,g}\le 0\ \ \beta_{\hat{t}}^{t,g}\le 0\ \ \theta_{\hat{t},\hat{p}}^{t,g}\le 0 & \forall\hat{t},\hat{p}
\end{aligned}
$$

By weak duality, $z(DRIOP(t,g))\le z(RIOP(t,g))$ for all t,g since $DRIOP(t,g)$ is a maximization problem and $RIOP(t,g)$ is a minimization problem. By enforcing $z^*(DRIOP(t,g))\ge 0$ for all t,g, we force the lower bounds on the real worst-case inventory levels to be greater than or equal to zero. By embedding $DRIOP(t,g)$ into the ROP, we obtain the following:

$$
\begin{aligned}
ROP\text{:}\ \min\ & \sum_{p=1}^{P}\sum_{t=1}^{T}\sum_{g=1}^{G} vrx_{t,p,g} \\
\text{s.t.}\ & w_{t,p,g} \leq \sum_{i=1}^{t} x_{i,p,g} - \sum_{i=1}^{t-1} w_{i,p,g} + inv_{0,p,g} && \forall t,p,g \\
& \sum_{p=1}^{P} w_{t,p,g} \geq \sum_{p=1}^{P} d_{t,p,g} && \forall t,g \\
& x_{t,p,g} \leq c_{p,g}^{pro} && \forall t,p,g \\
& K\gamma^{t,g} + \sum_{\hat{t}=1}^{T}(\beta_{\hat{t}}^{t,g} + \sum_{\hat{p}=1}^{P}\theta_{\hat{t},\hat{p}}^{t,g}) + \\
& \sum_{p=1}^{P}\left(\sum_{\hat{t}=1}^{t} x_{\hat{t},p,g} - \sum_{\hat{t}=1}^{t-1} w_{\hat{t},p,g} + inv_{0,p,g}\right) \geq 0 && \forall t,g \\
& \gamma^{t,g} + \beta_{\hat{t}}^{t,g} + \theta_{\hat{t},\hat{p}}^{t,g} \ \leq\ -x_{\hat{t},\hat{p},g} && \forall t,g,\ \forall \hat{t} \leq t,\ \forall \hat{p} \\
& \gamma^{t,g} + \beta_{\hat{t}}^{t,g} + \theta_{\hat{t},\hat{p}}^{t,g} \ \leq\ 0 && \forall t,g,\ \forall \hat{t} > t,\ \forall \hat{p} \\
& \sum_{i=1}^{t} x_{i,p,g} - \sum_{i=1}^{t-1} w_{i,p,g} + inv_{0,p,g} \leq c_{p,g}^{inv} && \forall t,p,g \\
& x_{t,p,g}, w_{t,p,g} \geq 0 && \forall t,p,g \\
& \gamma^{t,g} \leq 0 \quad \beta_{\hat{t}}^{t,g} \leq 0 \quad \theta_{\hat{t},\hat{p}}^{t,g} \leq 0 && \forall t,g,\hat{t},\hat{p}
\end{aligned}
$$

Proposition 3.2. *If the ROP is solved to optimality, then the following hold:*

- *$DRIOP(t,g)$ and $RIOP(t,g)$ are feasible for all t,g.*
- *$z(DRIOP(t,g)) = z(RIOP(t,g)) \geq 0$ for all t,g.*

Proof. When the ROP is solved to optimality, this implies the feasibility of the ROP. Therefore $DRIOP(t,g)$ is feasible for all t,g since the constraints of $DRIOP(t,g)$ are contained in those of the ROP. Furthermore, $z(DRIOP(t,g)) \geq 0$ for all t,g, since the opposite would imply that the ROP is infeasible. But $IOP(t,g)$ is feasible for all t,g since $\mathcal{U} \neq \emptyset$, hence all subproblems are primal and dual feasible. By strong duality, all subproblems are solved to optimality. □

3.5 Robust Methodology in Canonical Form

The approach we have described in the previous sections is in fact suitable for a general class of bi-level programs with following characteristics: (i) the outer problem is an LP, (ii) the inner problems are also LPs with common feasible regions. To clarify this connection, in this section we describe our approach using the canonical LP form. We start with the canonical formulation considering no uncertainty. We consider two subsets of constraints to distinguish the constraints that are directly affected by uncertainty (18c) from those that are not (18b).

$$\min \quad c^T x \tag{18a}$$

$$s.t. \quad Ax \geq b \tag{18b}$$

$$h^i(x) \geq 0 \;\; \forall i \in \mathcal{I} \tag{18c}$$

$$x \in \mathbb{R}^n_+ \tag{18d}$$

Then we introduce the uncertainty by reformulating constraints (18c) as inner problems (19c) which transforms the initial formulation to a bi-level problem:

$$\min \quad c^T x \tag{19a}$$

$$s.t. \quad Ax \geq b \tag{19b}$$

$$\min_{\xi \in \mathcal{U}} \{f^i(x, \xi)\} \geq 0 \;\; \forall i \in \mathcal{I} \tag{19c}$$

$$x \in \mathbb{R}^n_+ \tag{19d}$$

where[2]

$$\mathcal{U} = \{\xi \in \mathbb{R}^p_+ | H\xi \geq r\}, \tag{20a}$$

$$f^i(x, \xi) = x^T G^i \xi + x^T d^i + \xi^T (q^i) \;\; \forall i \in \mathcal{I}. \tag{20b}$$

In (20a), H is the constraint coefficient matrix for the inner problems; in (20b), G^i are the matrices; and in (20a), d^i and (q^i) are constant vectors of dimension n and p respectively. These help us represent the bi-linear form $f^i(x, \xi)$ in a compact manner. Note that we assumed that, when reformulated using the inner problem

[2] What are H and G? -LVS

variables, constraints (18c) will be written as the optimization problem (19c). The objective function in bilinear form is given in (20b) and is linear from the perspective of the inner problem since the x variables are fixed. Therefore the inner problems are convex optimization problems with linear objectives, which implies the optimality conditions of the inner problems are all linear. The inner problem i is:

$$\min \quad \left(x^T(G^i) + (q^i)^T\right)\xi \tag{21a}$$

$$s.t. \quad H\xi \geq r \quad Dual\ Var.:\ \ \lambda^i \geq 0 \tag{21b}$$

$$\xi \in \mathbb{R}^p_+ \tag{21c}$$

We first take the dual of the each inner problem i,

$$\max \quad r^T\lambda^i \tag{22a}$$

$$s.t. \quad H^T\lambda^i \leq x^T(G^i) + (q^i)^T \tag{22b}$$

$$\lambda^i \in R^m_+ \tag{22c}$$

and enforce the dual feasibility conditions in the outer problem. By weak duality, $\left(G^T x\right)^T \xi \geq r^T\lambda$, and the outer problem becomes

$$\min \quad c^T x \tag{23a}$$

$$s.t. \quad Ax \geq b \tag{23b}$$

$$x^T d^i + r^T\lambda^i \geq 0 \ \ \forall i \in \mathcal{I} \tag{23c}$$

$$H^T\lambda^i \leq x^T(G^i) + (q^i)^T \ \ \forall i \in \mathcal{I} \tag{23d}$$

$$\lambda^i \in R^m_+ \ \ \forall i \in \mathcal{I}, \ \ x \in R^n_+ \tag{23e}$$

3.6 Numerical Example

Consider an instance of the *ROP* with 2 plants, 7 time periods and 2 products. The production capacity is $c^{pro}_{p,g} = 5 \cdot 10^5$ and the inventory capacity is $c^{inv}_{p,g} = 10^6$. $K = 3$ interruptions are expected. Since we assumed a fixed rate for electricity, we solve the numerical instances of *ROP* with cost vectors where all the elements of the vector are identical to fixed rate. The production cost times the conversion factor is 1, i.e. $vr = 1$. However, the ROP is general enough to handle the case where the cost rate of electricity changes for each time period. The demand data and the optimal solution of the ROP are given in Table 2. Since we assumed the plants are identical, in Table 2 we observe identical production levels for both plants. Furthermore, we also observe identical production levels for all time periods, but this behavior is not observed for all data sets. We tested these solutions under all possible interruption scenarios and confirmed that, indeed, the production plan is feasible, i.e., no stock-outs occur. The behavior of the solution under two sample interruption scenarios is given in Tables 3 and 4. In the first scenario (Table 3), plant 1 is interrupted 3 times consecutively at the beginning of the horizon. In the second scenario (Table 4), plant 1 is interrupted in periods 5 and 7, and plant 2 is interrupted in period 6. In the interrupted time periods, the planned production levels for both products are replaced with 0 and the inventory pool levels for each time period/product are calculated using the updated production levels. The ROP provides a production plan such that under any interruption scenario, the inventory pool levels will be always non-negative. If the number of interruptions in a scenario is less than the anticipated level of interruptions, K, the ending inventory pool level will be positive unless there is at least one period where the demand is 0. Since we assumed identical plants, we have reported the robust solution that provides identical production levels for both plants. In general there can be multiple robust solutions and cases where the robust optimal model assigns different production levels to different plants depending on the constraints.

	Total Demand		**Solution**	Product 1		Product 2	
Periods	Prod. 1	Prod. 2	Period	Plant 1	Plant 2	Plant 1	Plant 2
1	78337	22086	1	63972	63972	4544	4544
2	113422	21967	2	63972	63972	4544	4544
3	172944	42249	3	63972	63972	4544	4544
4	122049	55444	4	63972	63972	4544	4544
5	147796	34464	5	63972	63972	4544	4544
6	140045	38057	6	63972	63972	4544	4544
7	129098	35711	7	63972	63972	4544	4544

Table 2: Demand Data and Solution

Product 1					
Period	Plant 1	Plant 2	Total Production	Inventory Pool	Demand
0	-	-	-	200000	-
1	0	63972	63972	185635	78337
2	0	63972	63972	136185	113422
3	0	63972	63972	27213	172944
4	63972	63972	127944	33108	122049
5	63972	63972	127944	13255	147796
6	63972	63972	127944	1154	140045
7	63972	63972	127944	0	129098

Product 2					
Period	Plant 1	Plant 2	Total Production	Inventory Pool	Demand
0	-	-	-	200000	-
1	0	4544	4544	182458	22086
2	0	4544	4544	165034	21967
3	0	4544	4544	127328	42249
4	4544	4544	9087	80972	55444
5	4544	4544	9087	55594	34464
6	4544	4544	9087	26624	38057
7	4544	4544	9087	0	35711

Table 3: Scenario 1: 3 consecutive interruptions at the beginning of the horizon at plant 1

Product 1					
Period	Plant 1	Plant 2	Total Production	Inventory Pool	Demand
0	-	-	-	200000	-
1	63972	63972	127944	249607	78337
2	63972	63972	127944	264129	113422
3	63972	63972	127944	219128	172944
4	63972	63972	127944	225023	122049
5	0	63972	63972	141199	147796
6	63972	0	63972	65126	140045
7	0	63972	63972	0	129098

Product 2					
Period	Plant 1	Plant 2	Total Production	Inventory Pool	Demand
0	-	-	-	200000	-
1	4544	4544	9087	187001	22086
2	4544	4544	9087	174121	21967
3	4544	4544	9087	140959	42249
4	4544	4544	9087	94602	55444
5	0	4544	4544	64682	34464
6	4544	0	4544	31168	38057
7	0	4544	4544	0	35711

Table 4: Scenario 2: Plant 1 is interrupted in periods 5 and 7, and plant 2 is interrupted in period 6

3.7 Robust Production Planning Heuristic

Depending on the magnitude of the input sets, the ROP may be a very large problem. For example, if there are 10 products, 10 plants, and 100 periods, the ROP has approximately one million variables and one million constraints. This problem may be difficult to solve exactly. Therefore, in this section we propose a heuristic that mimics the solution of the *ROP*. This heuristic applies only to the special case in which the plants are all identical. It also assumes that the production cost is fixed in all periods and the same at all plants. The heuristic evaluates the damage that would be caused by an interruption on that day, using each day's demand and its position in the horizon. Note that interruptions earlier in the horizon are more dangerous since early interruptions give the plants less time to build up their inventory. Therefore, the heuristic tries to "front-load" the production schedule to anticipate the most problematic interruptions.

We have assumed each product has is subject to individual production and inventory capacity constraints at each plant, however this might not need to be true in general. There might be conditions that effect the entire set of products such as joint inventory and capacity constraints:

$$\sum_{g\in\mathcal{G}} x_{t,p,g} \le c^{pro} \ \ \forall t,p \quad \text{(joint capacity)} \tag{24}$$

$$\sum_{g\in\mathcal{G}} inv_{t,p,g} \le c^{inv} \ \ \forall t,p \quad \text{(joint inventory)} \tag{25}$$

The formulation given for *ROP* does not contain constraints of these types, therefore one can effectively replace the given formulation with a set of formulations, each for a single product. In Algorithm 1, we describe a heuristic that can be applied when the outer problem is separable in terms of products.

Our heuristic consists of two steps. In the first step, the heuristic calculates, for each period t, the aggregate demand faced so far by all plants, represented by the non-decreasing sequence $\{a_t\}$. Next, for each period t, the heuristic calculates the number of available production slots in periods $1,\ldots,t$, assuming that the maximum possible number of interruptions occur by time t (at most one per period). This quantity is represented by the increasing sequence $\{s_t\}$. Up to time period $t = K$ there can be 1 interruption per time period hence $s_1 = 1(p-1), s_2 = 2(p-1),\ldots,s_K = K(p-1)$. For time periods $t > K$, $s_t = tp - K$ so the number of available production slots for each time period t can be characterized as $s_t = tp - \min\{t,K\}$. Next for each time period t, the heuristic calculates the "average production" required on each available plant to be able to satisfy running total demand a_t and characterize it as sequence $\{l_t\}$ where $l_t = \dfrac{a_t}{s_t}$. Then, the heuristic determines the period t^* with the largest l_t value, in an attempt to detect the period in which infeasibility is most likely, accounting for the timing of the interruptions and the quantity of the demands.

In the second step, the heuristic compares t^* to K. If $t^* \le K$ then it assigns the production levels as l_{t^*}

Algorithm 1 Robust Production Planning Heuristic: Single Product

Require: Indices, Sets and Parameters as defined in Table 1

STEP 1[3]

$a_t := \sum_{i=1}^{t} d_i \ \ \forall t \in \mathcal{T}$

$s_t := pt - \min\{t, K\} \ \ \forall t \in \mathcal{T}$

$l_t := \frac{a_t}{s_t} \ \ \forall t \in \mathcal{T}$

$t^* = arg \max_{t \in \mathcal{T}} l_t$

STEP 2

if $t^* \leq K$ **then**

$\forall t \leq t^*, \forall p \ x_{t,p} \leftarrow l_{t^*}$

$inv_0 \leftarrow inv_0 + t^*(p-1)l_{t^*} - a_{t^*}$[4]

if $\max_{t,p} x_{t,p} > c_p^{pro}$ **or** $\exists \hat{t} \in \{1, \ldots, t^*\}, \ \hat{p} \in \mathcal{P} \ \ s.t. \ \ inv_{\hat{t},\hat{p}} > c_{\hat{p}}^{inv}$ **then**

Declare Failure and HALT

end if

$K \leftarrow K - t^*$

$\mathcal{T} \leftarrow \{ord(t^*+1), \ldots, ord(T)\}$[5]

Go to STEP 1

else

$\forall t \leq t^*, \forall p \ x_{t,p} \leftarrow l_{t^*}$

$inv_0 \leftarrow inv_0 + (t^*p - K)l_{t^*} - a_{t^*}$

$K \leftarrow 0$

$\mathcal{T} \leftarrow \{ord(t^*+1), \ldots, ord(T)\}$

Update parameters in Table 1 according to $\mathcal{T}$[6]

Solve Standard Production Planning Problem (2) to obtain the optimal solution $y_{t,p} \ \forall t \in \mathcal{T}, p$

if $\max_{t,p} x_{t,p} > c_p^{pro}$ **or** $\exists \hat{t} \in \{1, \ldots, t^*\}, \ \hat{p} \in \mathcal{P} \ \ s.t. \ \ inv_{\hat{t},\hat{p}} > c_{\hat{p}}^{inv}$ **or** Problem (2) is Infeasible **then**

Declare Failure and HALT

end if

$\forall t > t^*, \forall p \ x_{t,p} \leftarrow y_{t,p}$

end if

at each plant for all time periods up to t^*. Then it updates initial inventory as $inv_0 + t^*(p-1)l_{t^*} - a_{t^*}$, which is calculated assuming t^* interruptions happened so far therefore it updates the number of interruptions as $K - t^*$. If $t^* > K$ then again it assigns the production levels to l_{t^*} at each plant for all time periods up to t^*. But this time, it updates the initial inventory as $inv_0 + (t^*p - K)l_{t^*} - a_{t^*}$. Finally it updates the set of time periods as $\mathcal{T} : \{ord(t^*+1), \ldots, (T)\}$, where $ord(t^*+1) = 1, \ldots, ord(T) = T - t^*$. After this step the heuristic goes back to step one and continues recursively until $K = 0$. When there are no interruptions left, it solves the standard production planning problem (2a-2g) and merge its solution to production plan. This heuristic takes at most K iterations, where each iteration consists of the aforementioned two steps. In step 2 of every iteration, the algorithm compares $x_{t,p}$ values to c_p^{pro} and inventory levels to c_p^{inv}. If $x_{t,p} > c_p^{pro}$ or $inv_{t,p} > c_p^{inv}$ then algorithms declares failure and terminates. Because of this termination rule, by construction every solution built by this heuristic is a feasible solution to ROP. The heuristic is summarized in Algorithm 1.

For the data given in Table 2, our heuristic calculates the exact optimal solution for the robust problem, as given in Table 2. Further numerical results are reported in §3.8, and again, the heuristic found the optimal solution in all instances tested. While our numerical results are hopeful on the optimality of this heuristic, we were unable to prove or disprove the optimality of the heuristic. Hence it stays as our conjecture that Algorithm 1 is actually an exact algorithm for finding the optimal solution of the ROP under the following conditions: (i) Products are not jointly constrained, i.e. ROP is separable in products, (ii) Plants are identical, (iii) Production costs and capacities are fixed throughout the horizon, (iv) Inventory capacities are fixed throughout the horizon. In all of the instances that we have tested, the heuristic either found the optimal solution or declared failure in the infeasible instances.

3.8 Computational Results

In this section, we report the results of a computational study designed to test the effectiveness of the robust model (ROP) as well as the heuristic presented in §3.7. We created 12 instance types, one for each combination of $T \in \{5, 10, 20, 40, 80, 160\}$ and $K \in \{0.2T, 0.4T\}$. For each instance type, we created 5 random instances in which the demands are generated as $d_t \sim unif(0, 3c^{pro})$. Costs and capacities are as defined in the numerical example given in §3.6. We modeled all of the problems using AMPL and solved them to optimality using the solver Gurobi 4.0. The heuristic was implemented in Matlab r2010a.

The results are reported in Table 5. The first column reports the instance name in the form $T.\mathrm{d}i$, where $i = 1, \ldots, 5$ is the random demand pattern. The next column reports the total demand for that instance. The table then lists, for the 20% interruption rate ($K = 0.2T$), the optimal objective value (found by Gurobi) and the associated CPU time and the objective value of the solution found by the heuristic (Algorithm 1) and the associated CPU time. The column labeled "Δ" reports the optimality gap, where $\Delta = |z(ALG1) - z(ROP)|$. The column labeled "Ψ" reports the ratio of total production to total demand (expressed as a percentage); that is, $\Psi = \dfrac{\text{Total Production}}{\text{Total Demand}}$. The last set of columns repeats this information for the 40% interruption rate.

Our heuristic found the optimal solution for the ROP for every instance (within the tolerance of $\Delta \leq 1$). Moreover, on average it executes an order of magnitude (10×) faster than solving the ROP directly with Gurobi. Note also from Table 5 that Ψ is greater for $K = 0.4T$ than for $K = 0.2T$; that is, as the number of interruptions increases, so does the total production. This result is also displayed in Figure 2, which plots Ψ for $K = 0.2T$ (lower point) and for $K = 0.4T$ (upper point) for each instance.

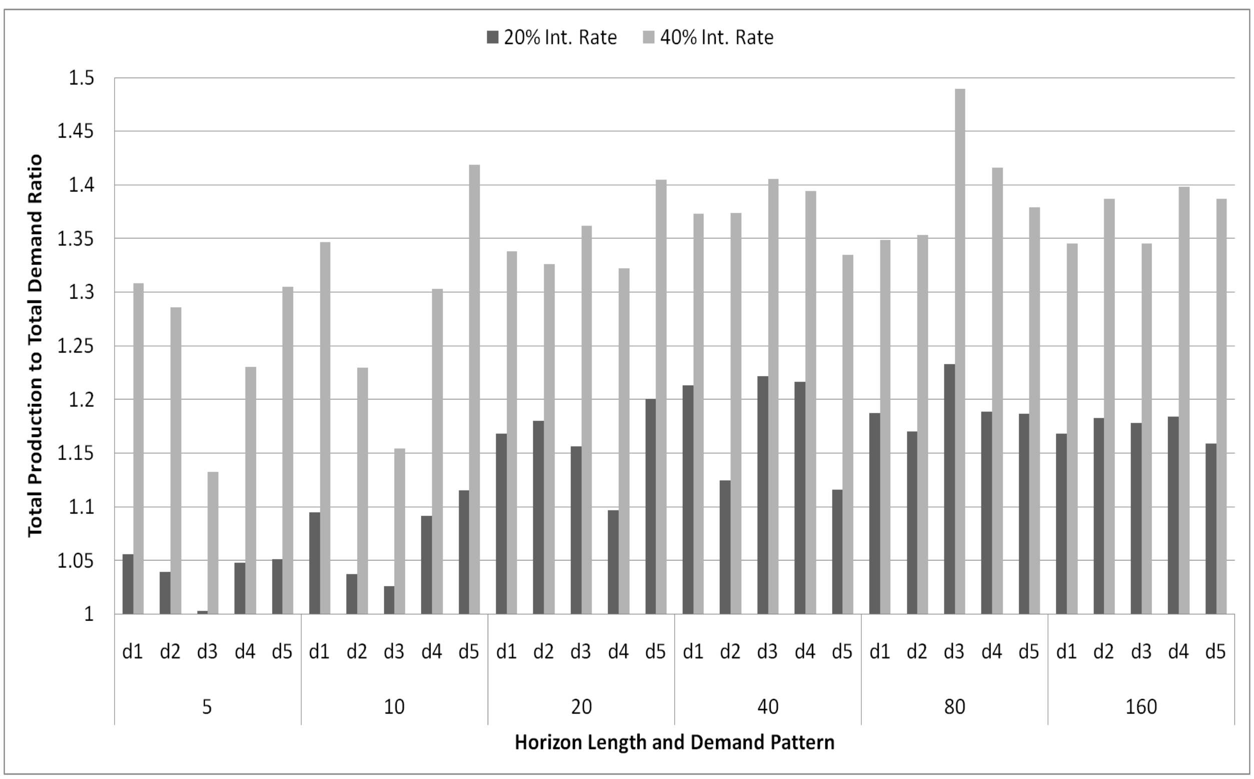

Figure 2: Total Production to Total Demand Ratio (Ψ) from Table 5

		20% Interruption Rate						40% Interruption Rate					
Problem	Total Demand (units)	Z(ALG1) (units)	Time (millisecs)	Z(ROP) (units)	Time (millisecs)	Δ	Ψ	Z(ALG1) (units)	Time (millisecs)	Z(ROP) (units)	Time (millisecs)	Δ	Ψ
5.d1	1,796,364	1,896,364.0	153.6	1,896,364	0	0	105.57%	2349731.0	7.4	2,349,731	0	0	130.80%
5.d2	2,026,226	2,105,192.0	1.6	2,105,192	0	0	103.90%	2605192.0	1	2,605,192	0	0	128.57%
5.d3	2,049,154	2,054,615.6	3.1	2,054,615	0	0.6	100.27%	2321216.0	0.4	2,321,215	0	1	113.28%
5.d4	2,082,395	2,182,395.0	0.4	2,182,395	0	0	104.80%	2562477.0	0.4	2,562,477	0	0	123.05%
5.d5	1,965,692	2,065,692.0	0.4	2,065,692	0	0	105.09%	2565692.0	0.5	2,565,692	0	0	130.52%
10.d1	3,106,557	3,401,842.2	4.1	3,401,842	0	0.2	109.51%	4182983.0	0.5	4,182,983	0	0	134.65%
10.d2	2,555,252	2,650,541.8	0.4	2,650,541	0	0.8	103.73%	3142691.3	0.5	3,142,691	0	0.3	122.99%
10.d3	2,611,483	2,679,425.6	0.4	2,679,425	0	0.6	102.60%	3014353.8	0.5	3,014,353	0	0.8	115.43%
10.d4	3,264,872	3,563,316.0	0.6	3,563,316	0	0	109.14%	4254910.0	3.7	4,254,910	0	0	130.32%
10.d5	3,187,867	3,555,612.0	0.6	3,555,612	10	0	111.54%	4522948.0	0.5	4,522,948	0	0	141.88%
20.d1	5,411,358	6,321,459.0	0.8	6,321,459	10	0	116.82%	7239097.0	1.3	7,239,097	10	0	133.78%
20.d2	4,492,461	5,301,452.0	0.6	5,301,452	10	0	118.01%	5959097.5	0.8	5,959,097	10	0.5	132.65%
20.d3	5,135,481	5,937,552.0	3.5	5,937,552	10	0	115.62%	6995207.0	0.8	6,995,207	10	0	136.21%
20.d4	5,360,054	5,877,101.5	0.6	5,877,101	10	0.5	109.65%	7087144.3	1.2	7,087,144	10	0.3	132.22%
20.d5	5,821,935	6,987,935.0	0.6	6,987,935	10	0	120.03%	8178779.2	1.1	8,178,779	10	0.2	140.48%
40.d1	10,151,455	12,311,710.5	1.7	12,311,710	40	0.5	121.28%	13936488.9	2.1	13,936,488	50	0.9	137.29%
40.d2	10,533,148	11,843,507.1	1.5	11,843,507	40	0.1	112.44%	14473102.0	1.1	14,473,101	40	1	137.41%
40.d3	10,224,667	12,492,212.0	1.1	12,492,212	50	0	122.18%	14368459.0	1.8	14,368,459	50	0	140.53%
40.d4	10,389,448	12,639,871.4	1.1	12,639,871	40	0.4	121.66%	14484549.9	1.5	14,484,549	30	0.9	139.42%
40.d5	10,252,456	11,444,351.6	1.1	11,444,351	50	0.6	111.63%	13686400.1	1.1	13,686,400	50	0.1	133.49%
80.d1	20,538,055	24,389,433.0	3.2	24,389,433	190	0	118.75%	27693071.0	4.8	27,693,071	190	0	134.84%
80.d2	20,586,039	24,093,313.8	1.9	24,093,313	180	0.8	117.04%	27865140.3	2.9	27,865,140	190	0.3	135.36%
80.d3	21,014,425	25,910,897.5	3	25,910,897	190	0.5	123.30%	31301283.0	1.9	31,301,283	180	0	148.95%
80.d4	18,607,001	22,115,780.0	2.5	22,115,779	180	1	118.86%	26346772.0	4.2	26,346,771	180	1	141.60%
80.d5	20,092,741	23,837,931.0	1.8	23,837,931	190	0	118.64%	27710980.0	1.7	27,710,979	180	1	137.92%
160.d1	38,474,902	44,946,580.7	6.4	44,946,580	860	0.7	116.82%	51751744.7	6.7	51,751,744	870	0.7	134.51%
160.d2	42,893,082	50,718,101.4	5.7	50,718,101	880	0.4	118.24%	59498000.3	11	59,498,000	950	0.3	138.71%
160.d3	43,904,907	51,734,708.0	3.5	51,734,708	890	0	117.83%	59081412.7	4	59,081,412	870	0.7	134.57%
160.d4	41,125,495	48,696,578.7	4.3	48,696,578	850	0.7	118.41%	57495417.0	2.6	57,495,417	870	0	139.80%
160.d5	38,225,164	44,297,082.3	3.3	44,297,082	890	0.3	115.88%	53016172.0	2.7	53,016,171	870	1	138.69%

Table 5: $T \in \{5, 10, 20, 40, 80, 160\}$, $K \in \{0.2T, 0.4T\}$, 2 Plants, 5 Demand Realizations, $inv_0 = 2 \cdot 10^5$

4 Production Modes

So far, we have assumed a very simple form for the interruptions and the firm's reaction to them. However, interruptions might have more complicated effects on production, at more than just the interrupted plant, and/or in more than just the interrupted period. In this section, we show how to embed operational rules that govern how the system operates during or after interruptions into the ROP. Each plant may operate in various *production modes* that are governed by the operational rules. In §3, we considered only the simplest possible operational rule (no production is allowed at interrupted plants) and only two production modes (interrupted and unaffected).

For example, consider the following operational rule: Once a plant recovers from an interruption, for one period the plant is in "recovery mode" in which its production rate is temporarily reduced. This rule may be imposed, for example, to give the interrupted plant time to ramp its production back to normal. (Such an operational rule is imposed for the air-separation plants that motivated this study.) This rule induces 3 production modes: interrupted, recovery and unaffected. The application of this logical rule for a given interruption scenario is depicted in Figure 3.

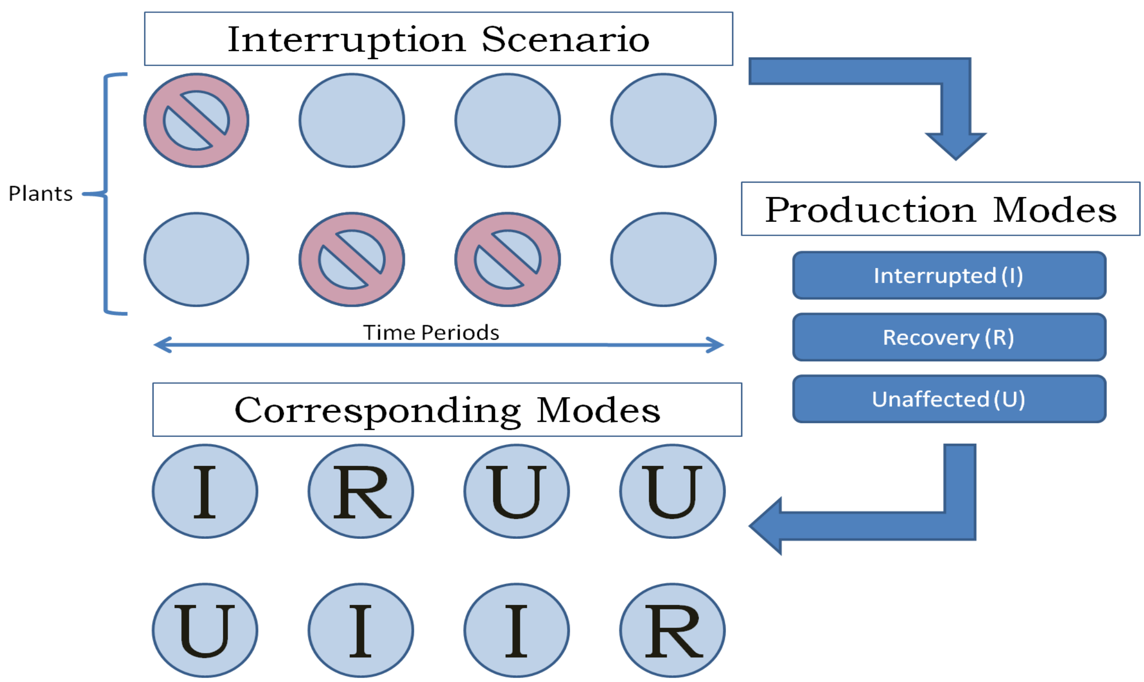

Figure 3: Interruptions and Corresponding Production Modes (⦸ denotes interruption)

We attempt to provide as general a framework as possible for modeling operational rules. Let $\mathcal{M}$ denote the set of production modes, and let

$$\xi_{m,t,p} = \begin{cases} 1, & \text{if plant } p \text{ is in production mode } m \text{ in period } t \\ 0, & \text{otherwise} \end{cases}$$

for $m \in \mathcal{M}$, $t \in \mathcal{T}$, $p \in \mathcal{P}$. The ξ variables indicate which production mode each plant is in, and they are a generalization of the ξ variables in 3.4. A production mode is defined by its effect on the production rate; in particular, let θ_m be the fraction of the normal production rate that a plant experiences when in production mode m. If $\theta_m = 0$, the production is completely interrupted; if $\theta_m = 1$, the plant is functioning normally; and if $0 < \theta_m < 1$, the plant is operating at a reduced rate. The θ parameters may be indexed by p and t if they are plant- and or period-dependent, but for ease of exposition we assume they depend only on m. Next, we relate the actual production to the planned production levels and the production modes, in a generalization of (6):

$$\bar{x}_{t,p,g} = \sum_{m \in \mathcal{M}} \theta_m \xi_{m,t,p} x_{t,p,g} \quad \forall t, p, g$$

Finally, the operational rules that govern the production modes must be represented as linear constraints on the ξ variables, and these constraints must be added to the uncertainty set $\mathcal{U}$.

To take an example, consider the "recovery mode" outlined above, and suppose that a plant in recovery mode experiences half its normal production rate. Then we have $\mathcal{M} = \{I, R, U\}$, $\theta_I = 0$, $\theta_R = 0.5$, and $\theta_U = 1$. The operational rules can be enforced by including the following constraints in $\mathcal{U}$:

$$\xi_{I,t,p} + \xi_{R,t,p} + \xi_{U,t,p} = 1 \quad \forall t, p \tag{26a}$$

$$\xi_{R,t,p} \geq \xi_{I,t-1,p} - \xi_{I,t,p} \quad \forall t, p \tag{26b}$$

$$\xi_{R,t,p} \leq \xi_{I,t-1,p} \quad \forall t, p \tag{26c}$$

$$\xi_{R,t,p} \leq 1 - \xi_{I,t,p} \quad \forall t, p \tag{26d}$$

Constraints (26a) require each plant to be in exactly one recovery mode in each period. Constraints (26b-26d) require plant p to be in recovery mode in period t if and only if it was interrupted in period $t-1$ but it is not interrupted in period t.

This approach for embedding operational rules into the ROP maintains the tractability of the problem. As long as the opponent's decision space ($\mathcal{U}$) does not depend on the outer problem variables, i.e., the production variables x, and the opponent's objective function can be expressed as a bilinear function of outer problem variables and inner problem variables, our methodology can continue to be used to combine the outer and inner problems into one linear program. In contrast, if the opponent's feasible region depends on the outer variables, then the resulting model will be nonlinear in general, although this may not be insurmountable If the inner problems can be reformulated as a set of optimality conditions, still one can cast the BLP as a single non-linear program and choose the appropriate solution method depending on the structure of this new non-linear programs. Moreover, if the inner problems are non-convex optimization problems, then this approach is not appropriate since the optimality conditions of the inner problems are necessary but not sufficient for inner problem optimality.

5 Conclusion

In this paper, we present a production planning framework for a rate-paying industrial production company whose production operations strongly depend on electricity. The problem we study is an operational-level, aggregate production and inventory planning problem with electricity supply uncertainty and deterministic demand. We assume that participation in an ILC provides a discounted and fixed rate to the production company, which effectively mitigates the negative impact of electricity price volatility but introduces supply uncertainty into the production system in the form of interruptions. Our robust production planning model accounts for this electricity supply uncertainty. In this model, we separate production decisions and interruption decisions, since production decisions belong to the industrial company while interruption decisions belong to the electricity retailer. The model can be solved using standard optimization software, but we also developed a heuristic that attempts to mimic the solution of the robust optimization model. In our computational experiment, our heuristic found the optimal solution for every instance, in approximately one-tenth the time, on average, as the exact approach.

The interruption uncertainty framework we describe allows different contract rules and operational rules to be embedded into the production planning problem simultaneously. As we discussed in §4, it is straightforward to embed operational procedures that companies may implement in the case of interruptions, such as limiting the production in post-interruption recovery or prohibiting production level increases in some periods. Similarly, our framework could be used under different types of ILCs, such as the pay-in-advance and pay-as-you-go reward schemes described by [2]. Moreover, information regarding the utility's optimal interruption dispatch behavior can be embedded into our Stackelberg-like production planning framework. However, the extent to which the theoretical results and computational performance presented above will be preserved under different ILC types or interruption dispatch behaviors is a topic for future study. Another important avenue for future study is to include demand uncertainty into the model.

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A Proof of Proposition 3.1

Note that for given t, g, $IOP(t, g)$ is:

$$\min_{\mathcal{U}} \left\{ \sum_{p=1}^{P} \overline{inv}_{t,p,g} \right\} \geq 0 \ \ \forall t, g \tag{27}$$

where

$$\mathcal{U} = \left\{ \xi \in \{0,1\}^{T \times P} \middle| \sum_{p=1}^{P} \sum_{t=1}^{T} \xi_{t,p} \leq K \ \ (^{*}), \ \ \sum_{p=1}^{P} \xi_{t,p} \leq 1 \ \ \forall t \ \ (^{**}) \right\} \tag{28}$$

Before proving Proposition 3.1 (total unimodularity of the constraint matrix of IOP), we first give three well known properties of TU matrices (see, e.g., [23]).

Lemma 1. *A matrix* $\mathbf{A}$ *is TU* $\iff$ $\mathbf{A^T}$ *is TU.*

Lemma 2. *A matrix* $\mathbf{A}$ *is TU* $\iff$ $[\mathbf{A} \quad \mathbf{I}]$ *is TU.*

Lemma 3. *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix whose rows can be partitioned into two disjoint sets* $\mathcal{B}$ *and* $\mathcal{C}$ *with the following properties:*

1. *Every column of* $\mathbf{A}$ *contains at most two non-zero entries;*

2. *Each entry is* 0, 1, *or* -1;

3. *If two non-zero entries in a column of* $\mathbf{A}$ *have the same sign, then the row of one is in* $\mathcal{B}$*, and the other in* $\mathcal{C}$*;*

4. *If two non-zero entries in a column of* $\mathbf{A}$ *have opposite signs, then the rows of both are in* $\mathcal{B}$*, or both in* $\mathcal{C}$*.*

Then $\mathbf{A}$ *is TU.*

Proof of Proposition 3.1. Consider the constraint matrix $\mathbf{A}$ defined by $\mathcal{U}$. It has the following form:

$$\mathbf{A} = \begin{bmatrix} & \mathbf{e}^T & \\ \mathbf{A}_T & \dots & \mathbf{I}_T \\ & \mathbf{I}_{T\times P} & \end{bmatrix} \tag{29}$$

where

$$\mathbf{e}^T = [1...1]_{T\times P}$$

The first row of $\mathbf{A}$, $\mathbf{e}^T$, captures the coefficients from the second inequality in $\mathcal{U}$. The second part, which is constructed by repeating $\mathbf{I}_T$ P times, captures the coefficients from the first set of inequalities in $\mathcal{U}$. Finally, $\mathbf{I}_{T\times P}$ is an identity matrix and captures the coefficients from the upper-bound inequalities for $\xi_{t,p}$. Now, define $\hat{\mathbf{A}}$ such that

$$\hat{\mathbf{A}} = \begin{bmatrix} & \mathbf{e}^T & \\ \mathbf{I}_T & \dots & \mathbf{I}_T \end{bmatrix} \tag{30}$$

Observe that $\hat{\mathbf{A}}$ is TU since it satisfies the conditions of Lemma 3. The first two conditions of Lemma 3 are satisfied trivially. For the third condition, all the nonzero elements of $\hat{\mathbf{A}}$ are positive, therefore one can construct an appropriate partition of the rows by putting the first row of $\hat{\mathbf{A}}$ in the set $\mathcal{B}$ and the rest of the rows in the set $\mathcal{C}$. The fourth condition doesn't apply since there are no negative elements in $\hat{\mathbf{A}}$. Now, by Lemma 1, $\hat{\mathbf{A}}^T$ is also TU. By Lemma 2, we can augment this matrix with $\mathbf{I}_{T\times P}$ to obtain $\mathbf{A}^T = [\hat{\mathbf{A}}^T \quad \mathbf{I}_{T\times P}]$ and still retain the TU property. Finally, by Lemma 1, we see that $\mathbf{A}$ is indeed TU. □

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