On the identification of the optimal partition for semidefinite optimization

Ali Mohammad-Nezhad and Tamás Terlaky

Department of Industrial and Systems Engineering, Lehigh University, USA

ISE Technical Report 17T-008

On the identification of the optimal partition for semidefinite optimization

Ali Mohammad-Nezhad[1] and Tamás Terlaky[1]

[1]Department of Industrial and Systems Engineering, Lehigh University, USA

June 14, 2017

Abstract

The concept of the optimal partition was originally introduced for linear optimization and linear complementarity problems and subsequently extended to semidefinite optimization. For linear optimization and sufficient linear complementarity problems, the optimal partition and a maximally complementary optimal solution can be identified in strongly polynomial time. In this paper, under no assumption on strict complementarity, we consider the identification of the optimal partition of semidefinite optimization for solutions on, or in a neighborhood of the central path. In contrast to linear optimization and linear complementarity problem, the optimal partition for semidefinite optimization cannot be identified exactly from an interior solution. Instead, we identify the sets of eigenvectors converging to an orthonormal bases for the optimal partition using the bounds for the magnitude of the eigenvalues. The magnitude of the eigenvalues of an interior solution is quantified using a condition number and an upper bound for the distance of an interior solution to the optimal set. We provide iteration complexity bound for the identification of the above sets of eigenvectors.

1 Introduction

Semidefinite optimization (SDO) is known as a generalization of linear optimization (LO), where the cone of symmetric positive semidefinite matrices substitutes for the nonnegative orthant. In SDO, one minimizes/maximizes the linear objective function

$$C \bullet X := \operatorname{tr}(CX),$$

where C and X are $n \times n$ symmetric matrices, over the intersection of the positive semidefinite cone and a set of affine constraints. Mathematically, an SDO problem is written as

$$(P) \qquad z_{p^*} := \min\left\{C \bullet X \mid A^i \bullet X = b_i, \quad i = 1, \ldots, m, \ X \succeq 0\right\},$$

where A^i for $i = 1, \ldots, m$ are $n \times n$ symmetric matrices, $b \in \mathbb{R}^m$, and $X \in \mathbb{S}^n_+$, where $\mathbb{S}^n_+$ denotes the cone of $n \times n$ positive semidefinite matrices. The dual SDO problem is given by

$$(D) \qquad z_{d^*} := \max\left\{b^T y \;\middle|\; \sum_{i=1}^m y_i A^i + S = C, \ S \succeq 0, \ y \in \mathbb{R}^m\right\}.$$

Let $\mathcal{P}$ and $\mathcal{D}$ denote the primal and dual feasible sets, respectively, as follows

$$\mathcal{P} := \left\{X \mid A^i \bullet X = b_i, \quad i = 1, \ldots, m, \ X \succeq 0\right\},$$

$$\mathcal{D} := \left\{(y, S) \;\middle|\; \sum_{i=1}^m y_i A^i + S = C, \ S \succeq 0\right\}.$$

In light of this notation, the primal and dual optimal sets are defined as

$$\mathcal{P}^* := \left\{X \mid X \in \mathcal{P},\ C \bullet X = z_{p^*}\right\},$$
$$\mathcal{D}^* := \Big\{(y,S) \;\Big|\; (y,S) \in \mathcal{D},\ b^T y = z_{d^*}\Big\}.$$

We assume that the matrices A^i for $i = 1, \dots, m$ are linearly independent. It is also assumed that the interior point condition holds, i.e., there exists $(X, y, S) \in \mathcal{P} \times \mathcal{D}$, where $X, S \succ 0$. The first assumption guarantees that y is uniquely determined for a given dual solution S, and the latter ensures that the primal and dual optimal sets are nonempty, and that strong duality holds. The interior point condition may be assumed w.l.o.g., since any SDO problem can be cast into a self-dual embedding format, for which the interior point condition always holds, see [5] for details.

SDO problems are frequently used in many applications, e.g., control theory, structural optimization, statistics, robust optimization, eigenvalue optimization, pattern recognition, and combinatorial optimization. Second-order conic optimization (SOCO) problems can be embedded in SDO formulation. See [28] for a detailed description of the problems. Analogous to LO, using interior point methods (IPMs), SDO problems can be solved in polynomial time, though they require significantly more computational effort per iteration. The extension of IPMs from LO to SDO was pioneered by Nesterov and Nemirovskii [17], and Alizadeh [1]. The main idea of primal-dual path following IPMs is to follow the central path, which is defined as the set of solutions of

$$\begin{aligned} A^i \bullet X &= b_i, \quad i = 1, \dots, m, \\ \sum_{i=1}^{m} A^i y_i + S &= C, \\ XS &= \mu I_n, \\ X, S &\succeq 0, \end{aligned} \tag{1}$$

where $XS = \mu I_n$ is called the centrality condition, and I_n denotes the identity matrix of size n. For a given $\mu > 0$, the central solution $(X(\mu), y(\mu), S(\mu))$ to this system exists and is uniquely defined under the interior point assumption and the linear independence of A^i for $i = 1, \dots, m$. For $0 \le \mu \le \bar{\mu}$, where $\bar{\mu} > 0$, the set of solutions of (1) is bounded, and thus the trajectory of the central solutions has limit points in the relative interior of the optimal set [7, 5, 16]. A proof was given by Halická et al. [11] stating that the central path converges to a maximally complementary optimal solution[1].

The analyticity and limiting behavior of the central path for SDO have been extensively studied in the literature. Luo, Sturm and Zhang [16] established the superlinear convergence of an IPM for SDO under the strict complementarity[2] assumption and a condition for the size of the neighborhood of the central path. The convergence of the central path to the so called analytic center of the optimal set was established by Luo, Sturm and Zhang [16] and de Klerk, Roos and Terlaky [5] under the strict complementarity condition. Goldfarb and Scheinberg [7] showed, under the strict complementarity and primal-dual nondegeneracy conditions, that the first order derivatives of the central path converge as $\mu \to 0$. However, the first order derivatives may be unbounded if strict complementarity fails to hold. Using the strict complementarity assumption only, Halická [10] showed the extension of the analyticity of the central path to $\mu = 0$.

In case of degeneracy, even for LO, the condition number of the Newton system of search directions goes to infinity, leading to ill-posed systems, during the final iterations of IPMs [9]. It would be helpful, like in LO and linear complementarity problem (LCP) [24, 13], if we could avoid this ill-conditioning, by switching over to a rounding procedure, when μ is sufficiently small. This motivates us to study the identification of the optimal partition.

[1] A primal-dual optimal solution (X^*, y^*, S^*) is called maximally complementary if $\operatorname{rank}(X^* + S^*)$ is maximal over all the optimal solutions.

[2] A primal-dual optimal solution (X^*, y^*, S^*) is called strictly complementary if $\operatorname{rank}(X^* + S^*) = n$.

The notion of the optimal partition was originally introduced for LO and LCPs. Ye [29] proposed a finite termination strategy for IPMs which generates a strictly complementary optimal solution from a primal-dual solution sufficiently close to the optimal set. Roos, Terlaky, and Vial [24] presented a rounding procedure which uses the optimal partition information to identify a strictly complementary optimal solution. Illés, Peng, Roos, and Terlaky [13] considered the identification of the optimal partition for sufficient LCPs and proposed a strongly polynomial rounding procedure to a maximally complementary optimal solution. The concept of the optimal partition was extended to SDO by Goldfarb and Scheinberg [7] and to general convex optimization by Yildirim [30]. Bonnans and Ramírez [2] established another algebraic definition of the optimal partition for SOCO. Peña and Roshchina [22] extended the idea of the complementarity partition for a linear system to a homogeneous convex conic system comprising of regular closed convex cones. Recently, Terlaky and Wang [27] have studied the identification of the optimal partition for SOCO.

In this paper, we consider the identification of the optimal partition for SDO. In contrast to LO and LCP, the optimal partition for SDO cannot be identified exactly. We identify the sets of eigenvectors converging to an orthonormal bases for the optimal partition using the bounds for the magnitude of the eigenvalues. The magnitude of the eigenvalues of an interior solution is quantified by using a condition number and an upper bound for the distance of an interior solution to the optimal set. The rest of this paper is organized as follows. In Section 2, we review the concepts of the optimal partition and complementarity. In Section 3, we analyze the magnitude of the eigenvalues of the solutions on the central path based on a condition number and error bound result for linear matrix inequalities (LMIs). The latter bound enables us to determine when the sets of eigenvectors converging to an orthonormal bases for the optimal partition can be identified. In Section 4, we extend the identification results to solutions in a neighborhood of the central path and provide iteration complexity bound for the identification of the above sets of eigenvectors. Finally, our conclusions are presented in Section 5.

Throughout this paper, any maximally complementary optimal solution is indicated by * superscript, but the limit point of the central path and the analytic center of the optimal set are denoted by (X^{**}, y^{**}, S^{**}) and (X^a, y^a, S^a), respectively. The subscript $[i]$ in our notation means the i^{th} largest component of a vector. For instance, $\lambda_{[i]}(X)$ denotes the i^{th} largest eigenvalue of X so that

$$\lambda_{[1]}(X) \geq \lambda_{[2]}(X) \geq \ldots \geq \lambda_{[n]}(X).$$

In particular, $\lambda_{\min}(X) := \lambda_{[n]}(X)$ and $\lambda_{\max}(X) := \lambda_{[1]}(X)$ stand for the minimum and maximum eigenvalues of X, respectively. Finally, by an orthogonal matrix we mean a square matrix whose columns are orthonormal, i.e., the columns are orthogonal, and they have unit length.

2 The optimal partition for SDO

Consider the optimality conditions for (P) and (D). Since the interior point condition holds, for optimality the KKT conditions [19] are necessary and sufficient for (P) and (D), which are written as

$$\begin{aligned} A^i \bullet X &= b_i, \quad i = 1, \ldots, m, \\ \sum_{i=1}^m A^i y_i + S &= C, \\ XS = 0, &\quad X, S \succeq 0, \end{aligned} \tag{2}$$

where $XS = 0$ is referred to as the complementarity condition. A solution (X, y, S) which satisfies $XS = 0$ is called complementary.

Definition 1 (Definition 2.7 in [4]). *Let $(X^*, y^*, S^*) \in \mathcal{P}^* \times \mathcal{D}^*$. Then (X^*, y^*, S^*) is a maximally complementary optimal pair if* $\operatorname{rank}(X^* + S^*)$ *is maximal over the optimal set. A maximally complementary pair (X^*, y^*, S^*) is strictly complementary if $X^* + S^* \succ 0$. The strict complementarity condition holds for an SDO problem if there exists a strictly complementary optimal solution.*

Note that strict complementarity may fail in SDO, i.e., an SDO problem might have no strictly complementary optimal solution. See [7, 4] for further details. A maximally complementary optimal pair can be equivalently defined as a primal-dual optimal solution in the relative interior of the optimal set. As a result, all $X^* \in \mathrm{ri}(\mathcal{P}^*)$ have the same range space. Analogously, all S^* have identical range spaces, where $(y^*, S^*) \in \mathrm{ri}(\mathcal{D}^*)$, see e.g., Lemma 2.3 in [4] or Lemma 3.1 in [7].

Let $\mathcal{B} := \mathcal{R}(X^*)$ and $\mathcal{N} := \mathcal{R}(S^*)$, where $\mathcal{R}(.)$ denotes the range space, and (X^*, y^*, S^*) is a maximally complementary optimal solution. We define $n_\mathcal{B} := \dim(\mathcal{B})$ and $n_\mathcal{N} := \dim(\mathcal{N})$. Then, it follows from the above equivalence, that $\mathcal{R}(X) \subseteq \mathcal{B}$ and $\mathcal{R}(S) \subseteq \mathcal{N}$ for all $(X, y, S) \in \mathcal{P}^* \times \mathcal{D}^*$. By the complementarity condition, the subspaces $\mathcal{B}$ and $\mathcal{N}$ are orthogonal, and this implies that $n_\mathcal{B} + n_\mathcal{N} \le n$. In case of strict complementarity, the subspaces $\mathcal{B}$ and $\mathcal{N}$ span $\mathbb{R}^n$. Otherwise, there exists a subspace $\mathcal{T}$, which is the orthogonal complement to $\mathcal{B}+\mathcal{N}$, i.e., $\mathbb{R}^n$ is partitioned into three mutually orthogonal subspaces $\mathcal{B}$, $\mathcal{N}$, and $\mathcal{T}$. In a similar manner, we define $n_\mathcal{T} := \dim(\mathcal{T})$.

Consider a maximally complementary optimal solution (X^*, y^*, S^*). By the complementarity condition, X^* and S^* commute, and thus they have the same eigenvector basis Q^*, i.e., we can represent X^* and S^* as

$$X^* = Q^* \Lambda(X^*)(Q^*)^T, \;\; S^* = Q^* \Lambda(S^*)(Q^*)^T,$$

where $\Lambda(X^*)$ and $\Lambda(S^*)$ are diagonal matrices containing the eigenvalues of X^* and S^*, respectively. Then we have

$$\mathcal{R}(X^*) = \mathcal{R}(Q^*\Lambda(X^*)), \quad \mathcal{R}(S^*) = \mathcal{R}(Q^*\Lambda(S^*)), \tag{3}$$

which implies that the range spaces are spanned by the eigenvectors associated with the positive eigenvalues. In particular, the eigenvectors corresponding to the positive eigenvalues of X^* can be chosen as an orthonormal basis for $\mathcal{B}$. In fact, any matrix with orthonormal columns which span $\mathcal{B}$ would be an orthonormal basis for $\mathcal{B}$. Analogously, we can choose the eigenvectors corresponding to the positive eigenvalues of S^* as an orthonormal basis for $\mathcal{N}$.

Definition 2. *The partition ($\mathcal{B}$, $\mathcal{N}$,$\mathcal{T}$) of $\mathbb{R}^n$ is called the optimal partition of an SDO problem.*

Let $Q := [Q_\mathcal{B}, Q_\mathcal{T}, Q_\mathcal{N}]$ denote an orthonormal bases for $\mathcal{B}$, $\mathcal{N}$, and $\mathcal{T}$, respectively. Now, the following theorem is in order.

Theorem 1 (Theorem 2.7 in [4])**.** *For every primal-dual optimal solution $(X, y, S) \in \mathcal{P}^* \times \mathcal{D}^*$ we can represent X and S as*

$$X = Q_\mathcal{B} U_X Q_\mathcal{B}^T, \;\; S = Q_\mathcal{N} U_S Q_\mathcal{N}^T,$$

where U_X and U_S are symmetric $n_\mathcal{B} \times n_\mathcal{B}$ and $n_\mathcal{N} \times n_\mathcal{N}$ positive semidefinite matrices, respectively. If $n_\mathcal{B} > 0$ and $X^* \in \mathrm{ri}(\mathcal{P}^*)$, then there exists $U_{X^*} \succ 0$. Similarly, if $n_\mathcal{N} > 0$ and $S^* \in \mathrm{ri}(\mathcal{D}^*)$, then there exists $U_{S^*} \succ 0$.

Notice the necessity of the condition $n_\mathcal{B} > 0$ or $n_\mathcal{N} > 0$ in Theorem 1. For instance, if $n_\mathcal{B} = 0$, then we have $\mathcal{P}^* = \mathrm{ri}(\mathcal{P}^*) = \{0\}$, which implies $U_{X^*} = 0$.

Remark 1. *By the interior point condition, at least one of $n_\mathcal{B}$ or $n_\mathcal{N}$ has to be positive. In fact, if $X^* = 0$ is the unique primal optimal solution of (P), then any dual feasible solution is also dual optimal. Therefore, by the interior point condition, there exists a dual optimal solution (y^*, S^*) where S^* is positive definite. Similarly, for a unique dual optimal solution (y^*, S^*) with $S^* = 0$ there exists a primal optimal solution X^* which is positive definite. Consequently, when either $n_\mathcal{B} = 0$ or $n_\mathcal{N} = 0$ holds, then there exists an optimal solution which is strictly complementary.*

An orthogonal transformation of $(X^*, y^*, S^*) \in \mathrm{ri}(\mathcal{P}^* \times \mathcal{D}^*)$ with respect to Q reveals the optimal partition as

$$Q^T X^* Q = \begin{bmatrix} U_{X^*} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q^T S^* Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & U_{S^*} \end{bmatrix},$$

where $U_{X^*} \succ 0$ and $U_{S^*} \succ 0$ if $n_{\mathcal{B}}, n_{\mathcal{N}} > 0$. As a result of Theorem 1 we have

$$\begin{aligned} Q_{\mathcal{T}\cup\mathcal{N}}^T X Q_{\mathcal{T}\cup\mathcal{N}} &= 0, \quad \forall\ X \in \mathcal{P}^*, \\ Q_{\mathcal{B}\cup\mathcal{T}}^T S Q_{\mathcal{B}\cup\mathcal{T}} &= 0, \quad \forall\ (y, S) \in \mathcal{D}^*, \end{aligned}$$

where $Q_{\mathcal{T}\cup\mathcal{N}} := [Q_{\mathcal{T}}\ Q_{\mathcal{N}}]$, and $Q_{\mathcal{B}\cup\mathcal{T}} := [Q_{\mathcal{B}}\ Q_{\mathcal{T}}]$.

Let $\Gamma_{\mathcal{B}}$ and $\Gamma_{\mathcal{N}}$ denote the set of all orthonormal bases for $\mathcal{B}$ and $\mathcal{N}$, respectively. The following lemma is in order.

Lemma 1. *The sets $\Gamma_{\mathcal{B}}$ and $\Gamma_{\mathcal{N}}$ are compact.*

Proof. If $\mathcal{B} = \{0\}$, then the lemma holds trivially. Hence, we can assume that $\mathcal{B} \neq \{0\}$. Then it is known that for a given subspace $\mathcal{B}$, any two orthonormal bases $Q_{\mathcal{B}}$ and $\bar{Q}_{\mathcal{B}}$ are related by $Q_{\mathcal{B}} U = \bar{Q}_{\mathcal{B}}$ for some orthogonal matrix $U \in \mathbb{R}^{n_{\mathcal{B}} \times n_{\mathcal{B}}}$, see e.g., Lemma 2.4 in [4]. The result follows by noting that the set of orthogonal matrices is compact. The compactness of $\Gamma_{\mathcal{N}}$ follows analogously. □

3 On the identification of the optimal partition along the central path

Recall that the central path for (P) and (D) is defined by (1), and assume that $Q_{\mathcal{B}}$ and $Q_{\mathcal{N}}$ are known. To derive bounds for the magnitudes of the eigenvalues of $X(\mu)$ and $S(\mu)$ on the central path as $\mu \to 0$, we define some condition numbers as

$$\sigma_{\mathcal{B}} := \max_{X \in \mathcal{P}^*} \lambda_{\min}(Q_{\mathcal{B}}^T X Q_{\mathcal{B}}) = \max_{\bar{Q}_{\mathcal{B}} \in \Gamma_{\mathcal{B}}} \max_{X \in \mathcal{P}^*} \lambda_{\min}(\bar{Q}_{\mathcal{B}}^T X \bar{Q}_{\mathcal{B}}), \tag{4}$$

$$\sigma_{\mathcal{N}} := \max_{(y,S) \in \mathcal{D}^*} \lambda_{\min}(Q_{\mathcal{N}}^T S Q_{\mathcal{N}}) = \max_{\bar{Q}_{\mathcal{N}} \in \Gamma_{\mathcal{N}}} \max_{(y,S) \in \mathcal{D}^*} \lambda_{\min}(\bar{Q}_{\mathcal{N}}^T S \bar{Q}_{\mathcal{N}}), \tag{5}$$

$$\sigma := \min\{\sigma_{\mathcal{B}}, \sigma_{\mathcal{N}}\}. \tag{6}$$

Let us define $\sigma_{\mathcal{B}} := \infty$ if $n_{\mathcal{B}} = 0$, and $\sigma_{\mathcal{N}} := \infty$ if $n_{\mathcal{N}} = 0$.

Lemma 2. *The condition number σ is positive.*

Proof. By the interior point assumption, $\mathcal{P}^* \times \mathcal{D}^*$ is nonempty and compact, see e.g., Lemma 3.2 in [7]. Thus, σ is well-defined by Remark 1. Assume that $n_{\mathcal{B}} > 0$. Then there exists $X \in \mathcal{P}^*$ so that $\lambda_{\min}(Q_{\mathcal{B}}^T X Q_{\mathcal{B}}) > 0$. By the compactness of $\mathcal{P}^*$ and the continuity of the eigenvalues, there exists $\bar{X} \in \mathcal{P}^*$ so that

$$\max_{X \in \mathcal{P}^*} \lambda_{\min}(Q_{\mathcal{B}}^T X Q_{\mathcal{B}}) = \lambda_{\min}(Q_{\mathcal{B}}^T \bar{X} Q_{\mathcal{B}}) \geq \lambda_{\min}(Q_{\mathcal{B}}^T X Q_{\mathcal{B}}) > 0,$$

which implies that $\sigma_{\mathcal{B}} > 0$. A similar argument can be made to show that $\sigma_{\mathcal{N}} > 0$ if $n_{\mathcal{N}} > 0$. Consequently, it holds that $\sigma > 0$. □

Remark 2. *In Appendix A, we provide a positive lower bound for the condition number σ. See Lemma 9 for a detailed discussion.*

Example 1. *Consider the following SDO problem from [8]:*

$$A^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; A^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix},$$
$$b^T = [1,\ 0,\ 0].$$

The problem satisfies the strict complementarity condition, and thus $\mathcal{T} = \{0\}$. The primal optimal face is given by

$$X^*_\theta = \begin{bmatrix} 1 & 2(\theta-1) & 2(\theta-1) \\ 2(\theta-1) & 4(1-\theta) & 4(1-\theta) \\ 2(\theta-1) & 4(1-\theta) & 4(1-\theta) \end{bmatrix}, \quad 0 \le \theta \le 1, \tag{7}$$

and the unique dual optimal solution is

$$S^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad y^* = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

An orthonormal bases for the optimal partition of the problem is given by

$$Q_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix}, \quad Q_{\mathcal{N}} = \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}.$$

Since the dual optimal solution is unique, it immediately follows from (5) *that*

$$\sigma_{\mathcal{N}} = \lambda_{\min}(Q_{\mathcal{N}}^T S^* Q_{\mathcal{N}}) = 2.$$

Given the primal optimal face (7)*, the condition number $\sigma_{\mathcal{B}}$ can be computed by solving*

$$\begin{aligned} \sigma_{\mathcal{B}} = \max_{0\le\theta\le1} \lambda_{\min}(Q_{\mathcal{B}}^T X^*_\theta Q_{\mathcal{B}}) &= \max_{0\le\theta\le1} \lambda_{\min}\left(\begin{bmatrix} 1 & 2\sqrt{2}(\theta-1) \\ 2\sqrt{2}(\theta-1) & 8(1-\theta) \end{bmatrix}\right) \\ &= \max_{0\le\theta\le1} -\frac{1}{2}\sqrt{96\theta^2 - 176\theta + 81} - 4\theta + \frac{9}{2}, \end{aligned}$$

which attains the maximum at $\theta^* = \frac{5}{6}$, and so $\sigma_{\mathcal{B}} = \frac{2}{3}$. Thus, we get

$$\sigma = \min\left\{\frac{2}{3}, 2\right\} = \frac{2}{3}.$$

Consider the orthogonal transformation of a central solution $(X(\mu), y(\mu), S(\mu))$ with respect to Q denoted by

$$\hat{X}(\mu) := \begin{bmatrix} \hat{X}_{\mathcal{B}}(\mu) & \hat{X}_{\mathcal{BT}}(\mu) & \hat{X}_{\mathcal{BN}}(\mu) \\ \hat{X}_{\mathcal{TB}}(\mu) & \hat{X}_{\mathcal{T}}(\mu) & \hat{X}_{\mathcal{TN}}(\mu) \\ \hat{X}_{\mathcal{NB}}(\mu) & \hat{X}_{\mathcal{NT}}(\mu) & \hat{X}_{\mathcal{N}}(\mu) \end{bmatrix}, \; \hat{S}(\mu) := \begin{bmatrix} \hat{S}_{\mathcal{B}}(\mu) & \hat{S}_{\mathcal{BT}}(\mu) & \hat{S}_{\mathcal{BN}}(\mu) \\ \hat{S}_{\mathcal{TB}}(\mu) & \hat{S}_{\mathcal{T}}(\mu) & \hat{S}_{\mathcal{TN}}(\mu) \\ \hat{S}_{\mathcal{NB}}(\mu) & \hat{S}_{\mathcal{NT}}(\mu) & \hat{S}_{\mathcal{N}}(\mu) \end{bmatrix}, \tag{8}$$

where $\hat{X}(\mu) := Q^T X(\mu) Q$ and $\hat{S}(\mu) := Q^T S(\mu) Q$. In what follows, we resort to an error bound result for an LMI system from Theorem 3.3 in [26], see also Lemma 10 in Appendix B. Lemma 3 specifies an upper bound for the distance of a central solution to the optimal set.

Lemma 3. *Let $\big(X(\mu), y(\mu), S(\mu)\big)$ be a central solution with $n\mu \le 1$. Then there exists $(X, y, S) \in \mathcal{P}^* \times \mathcal{D}^*$ so that*

$$\big\|X(\mu) - X\big\| \le c(n\mu)^\gamma, \qquad \big\|S(\mu) - S\big\| \le c(n\mu)^\gamma, \tag{9}$$

where $\|.\|$ stands for the Frobenius norm of a matrix, $\gamma = 2^{-d_s}$, in which d_s denotes the degree of singularity[3] *of the minimal subspace containing the optimal set, and c is a positive condition number.*

Proof. The bound can be established easily by applying the error bound for an LMI system as stated in Lemma 10. Note that the optimality conditions for an SDO problem are equivalent to

$$\begin{aligned} &A^i \bullet X = b_i, \quad i = 1, \ldots, m, \\ &\sum_{i=1}^m A^i y_i + S = C, \\ &C \bullet X - b^T y \le 0, \quad X, S \succeq 0. \end{aligned} \tag{10}$$

As defined by (1), the set of central solutions $(X(\mu), y(\mu), S(\mu))$ for $0 < \mu \le \frac{1}{n}$ is bounded, see Lemma 3.2 in [4]. Furthermore, the amount of violation of the third linear inequality in (10) at a given central solution $(X(\mu), y(\mu), S(\mu))$ is equal to $n\mu$, see Section 4 in [26]. Therefore, from Lemma 10 and the compactness of the optimal set it follows the existence of $(X, y, S) \in \mathcal{P}^* \times \mathcal{D}^*$ so that

$$\sqrt{\big\|X(\mu) - X\big\|^2 + \big\|y(\mu) - y\big\|^2 + \big\|S(\mu) - S\big\|^2} \le c(n\mu)^\gamma,$$

where $\gamma = 2^{-d_s}$. This completes the proof. ☐

From Lemma 11 in Appendix B we can get a nontrivial upper bound $n-1$ for the degree of singularity. Therefore, we have $\gamma \ge \frac{1}{2^{n-1}}$ for $n \ge 2$.

Remark 3. *For the special case $\mathcal{T} = \{0\}$, the degree of singularity is at most 1 [26]. For instance, this special case happens when we embed an LO problem in SDO. Then Lemma 3 gives an upper bound $\mathcal{O}(\sqrt{n\mu})$ for the distance of a central solution to the optimal set. However, a direct application of the Hoffman error bound*[4] *to the linear system of the optimality conditions results in the upper bound $\mathcal{O}(n\mu)$.*

Since the central path converges to a maximally complementary optimal solution, from the orthogonal transformation in (8) we have

$$\lim_{\mu \to 0} \hat{X}_{\mathcal{B}}(\mu) = U_{X^{**}}, \quad \text{and} \quad \lim_{\mu \to 0} \hat{S}_{\mathcal{N}}(\mu) = U_{S^{**}},$$

and

$$\lim\nolimits_{\mu \to 0} Q_{\mathcal{T}\cup\mathcal{N}}^T X(\mu) Q_{\mathcal{T}\cup\mathcal{N}} = 0, \qquad \lim\nolimits_{\mu \to 0} Q_{\mathcal{B}\cup\mathcal{T}}^T S(\mu) Q_{\mathcal{B}\cup\mathcal{T}} = 0,$$

where $\hat{X}_{\mathcal{B}}(\mu) = Q_{\mathcal{B}}^T X(\mu) Q_{\mathcal{B}}$, $\hat{S}_{\mathcal{N}}(\mu) = Q_{\mathcal{N}}^T S(\mu) Q_{\mathcal{N}}$, and (X^{**}, y^{**}, S^{**}) denotes the limit point of the central path. The following lemma establishes upper bounds for the vanishing blocks of $\hat{X}(\mu)$ and $\hat{S}(\mu)$. We can assume w.l.o.g. that both $n_{\mathcal{B}}$ and $n_{\mathcal{N}}$ are positive. For instance, if $n_{\mathcal{B}} = 0$, then we can interpret that the bounds for $\mathcal{B}$ do not exist.

Lemma 4. *Assume that μ is sufficiently small so that $n\mu \le 1$. Then we have*

$$\begin{aligned} &\operatorname{tr}(\hat{X}_{\mathcal{N}}(\mu)) \le \frac{n\mu}{\sigma}, && \operatorname{tr}(\hat{S}_{\mathcal{B}}(\mu)) \le \frac{n\mu}{\sigma}, \\ &\big\|Q_{\mathcal{T}\cup\mathcal{N}}^T X(\mu) Q_{\mathcal{T}\cup\mathcal{N}}\big\| \le c(n\mu)^\gamma, && \big\|Q_{\mathcal{B}\cup\mathcal{T}}^T S(\mu) Q_{\mathcal{B}\cup\mathcal{T}}\big\| \le c(n\mu)^\gamma. \end{aligned}$$

[3] The degree of singularity [26] is defined as the minimum number of facial reduction steps to get the minimal face of the positive semidefinite cone which contains the optimal set.

[4] See e.g., Theorem 9 in [20].

Proof. By the compactness of $\mathcal{P}^* \times \mathcal{D}^*$ and the continuity of the eigenvalues there exists $\bar{X} \in \mathcal{P}^*$ so that $\sigma_{\mathcal{B}} = \lambda_{\min}(Q_{\mathcal{B}}^T \bar{X} Q_{\mathcal{B}})$ as defined in (4). Analogously, it follows from (5) that $\sigma_{\mathcal{N}} = \lambda_{\min}(Q_{\mathcal{N}}^T \bar{S} Q_{\mathcal{N}})$ for some $(\bar{y}, \bar{S}) \in \mathcal{D}^*$. Since $\sigma = \min\{\sigma_{\mathcal{B}}, \sigma_{\mathcal{N}}\}$, then we should have $(\bar{X}, \bar{y}, \bar{S}) \in \mathcal{P}^* \times \mathcal{D}^*$ so that

$$\lambda_{\min}(U_{\bar{X}}) \geq \sigma, \quad \lambda_{\min}(U_{\bar{S}}) \geq \sigma, \tag{11}$$

where $U_{\bar{X}} = Q_{\mathcal{B}}^T \bar{X} Q_{\mathcal{B}}$ and $U_{\bar{S}} = Q_{\mathcal{N}}^T \bar{S} Q_{\mathcal{N}}$. Recall from the optimality conditions that

$$(X(\mu) - \bar{X}) \bullet (S(\mu) - \bar{S}) = 0,$$

which by (1) gives

$$X(\mu) \bullet \bar{S} + \bar{X} \bullet S(\mu) = n\mu.$$

Since the inner product is invariant with respect to an orthogonal transformation, we get

$$X(\mu) \bullet \bar{S} + \bar{X} \bullet S(\mu) = \hat{X}_{\mathcal{N}}(\mu) \bullet U_{\bar{S}} + U_{\bar{X}} \bullet \hat{S}_{\mathcal{B}}(\mu) = n\mu,$$

where $\hat{S}_{\mathcal{B}}(\mu) = Q_{\mathcal{B}}^T S(\mu) Q_{\mathcal{B}}$ and $\hat{X}_{\mathcal{N}}(\mu) = Q_{\mathcal{N}}^T X(\mu) Q_{\mathcal{N}}$. Therefore, the positive definiteness of $\hat{X}_{\mathcal{N}}(\mu)$ gives rise to $\hat{X}_{\mathcal{N}}(\mu) \bullet U_{\bar{S}} \leq n\mu$. Furthermore, from the inequality $\lambda_{\min}(U_{\bar{S}}) \operatorname{tr}(\hat{X}_{\mathcal{N}}(\mu)) \leq \hat{X}_{\mathcal{N}}(\mu) \bullet U_{\bar{S}}$, it immediately follows that

$$\lambda_{\min}(U_{\bar{S}}) \operatorname{tr}(\hat{X}_{\mathcal{N}}(\mu)) \leq n\mu,$$

which by the lower bounds (11) gives

$$\operatorname{tr}(\hat{X}_{\mathcal{N}}(\mu)) \leq \frac{n\mu}{\sigma}.$$

In a similar manner, it follows from $\hat{S}_{\mathcal{B}}(\mu) \succ 0$ that

$$\operatorname{tr}(\hat{S}_{\mathcal{B}}(\mu)) \leq \frac{n\mu}{\sigma}.$$

From Lemma 3, there exists $(X, y, S) \in \mathcal{P}^* \times \mathcal{D}^*$ so that (9) holds. Recall from Theorem 1 that X can be represented as $Q_{\mathcal{B}} U_X Q_{\mathcal{B}}^T$ where $U_X \succeq 0$. Thus, we have

$$\left\| Q_{\mathcal{T}\cup\mathcal{N}}^T X(\mu) Q_{\mathcal{T}\cup\mathcal{N}} \right\| = \left\| \begin{bmatrix} \hat{X}_{\mathcal{T}}(\mu) & \hat{X}_{\mathcal{T}\mathcal{N}}(\mu) \\ \hat{X}_{\mathcal{N}\mathcal{T}}(\mu) & \hat{X}_{\mathcal{N}}(\mu) \end{bmatrix} \right\| \leq \left\| X(\mu) - X \right\| \leq c(n\mu)^{\gamma},$$

and

$$\left\| Q_{\mathcal{B}\cup\mathcal{T}}^T S(\mu) Q_{\mathcal{B}\cup\mathcal{T}} \right\| = \left\| \begin{bmatrix} \hat{S}_{\mathcal{B}}(\mu) & \hat{S}_{\mathcal{B}\mathcal{T}}(\mu) \\ \hat{S}_{\mathcal{T}\mathcal{B}}(\mu) & \hat{S}_{\mathcal{T}}(\mu) \end{bmatrix} \right\| \leq \left\| S(\mu) - S \right\| \leq c(n\mu)^{\gamma},$$

which completes the proof. □

Note that the eigenvalues and eigenvectors of a central solution vary continuously as $\mu \to 0$. In other words, compared to its counterpart in LO and LCP, the optimal partition of $\mathbb{R}^n$ to the three subspaces $\mathcal{B}$, $\mathcal{N}$, and $\mathcal{T}$ cannot be identified exactly from a central solution sufficiently close to the optimal set. However, we show in Theorems 2 and 3 that the sets of eigenvectors converging to an orthonormal bases for $\mathcal{B}$, $\mathcal{N}$, and $\mathcal{T}$ can be identified, when μ is sufficiently small.

Lemma 5 (Theorem 2.3.8 in [14] or Theorem 4.5 in [25])**.** *Let X be an $n\times n$ symmetric matrix and $Y \in \mathbb{R}^{n\times k}$ be a matrix with orthonormal columns. Then we have*

$$\begin{aligned} \lambda_{[n-k+1]}(X) + \ldots + \lambda_{[n]}(X) = \min_{Y} \quad & \operatorname{tr}(Y^T X Y), \\ \text{s.t.} \quad & Y^T Y = I_k. \end{aligned}$$

Theorem 2. *For a central solution $(X(\mu), y(\mu), S(\mu))$ with $n\mu \le 1$, it holds that:*

1. *For $i = 1, \dots, n_{\mathcal{B}}$ we have*
$$\lambda_{[n-i+1]}(S(\mu)) \le \frac{n\mu}{\sigma}, \qquad \lambda_{[i]}(X(\mu)) \ge \frac{\sigma}{n}. \tag{12}$$

2. *For $i = 1, \dots, n_{\mathcal{N}}$ we have*
$$\lambda_{[i]}(S(\mu)) \ge \frac{\sigma}{n}, \qquad \lambda_{[n-i+1]}(X(\mu)) \le \frac{n\mu}{\sigma}. \tag{13}$$

Furthermore, we have

$$\begin{aligned} \lambda_{[n-i+1]}(X(\mu)) &\le c\sqrt{n}(n\mu)^{\gamma}, \qquad \lambda_{[i]}(S(\mu)) \ge \frac{\mu}{c\sqrt{n}(n\mu)^{\gamma}}, \ \ i = 1, \dots, n_{\mathcal{N}} + n_{\mathcal{T}}, \\ \lambda_{[n-i+1]}(S(\mu)) &\le c\sqrt{n}(n\mu)^{\gamma}, \qquad \lambda_{[i]}(X(\mu)) \ge \frac{\mu}{c\sqrt{n}(n\mu)^{\gamma}}, \ \ i = 1, \dots, n_{\mathcal{B}} + n_{\mathcal{T}}. \end{aligned} \tag{14}$$

If $n_{\mathcal{T}} > 0$, then we have $c \ge 1$, and

$$\frac{1}{2^{n-1}} \le \gamma \le \frac{1}{2}.$$

Proof. Note that $\hat{S}_{\mathcal{B}}(\mu) = Q_{\mathcal{B}}^T S(\mu) Q_{\mathcal{B}}$ and $\hat{X}_{\mathcal{N}}(\mu) = Q_{\mathcal{N}}^T X(\mu) Q_{\mathcal{N}}$ as defined in (8). Then it follows from Lemma 5 that

$$\begin{aligned} \lambda_{[n-n_{\mathcal{B}}+1]}(S(\mu)) + \ldots + \lambda_{[n]}(S(\mu)) &\le \operatorname{tr}(\hat{S}_{\mathcal{B}}(\mu)) \le \frac{n\mu}{\sigma}, \\ \lambda_{[n-n_{\mathcal{N}}+1]}(X(\mu)) + \ldots + \lambda_{[n]}(X(\mu)) &\le \operatorname{tr}(\hat{X}_{\mathcal{N}}(\mu)) \le \frac{n\mu}{\sigma}. \end{aligned}$$

Therefore, noting that $\lambda_{\min}(X(\mu)), \lambda_{\min}(S(\mu)) > 0$, we get

$$\begin{aligned} \lambda_{[n-i+1]}(S(\mu)) &\le \frac{n\mu}{\sigma}, \qquad i = 1, \dots, n_{\mathcal{B}}, \\ \lambda_{[n-i+1]}(X(\mu)) &\le \frac{n\mu}{\sigma}, \qquad i = 1, \dots, n_{\mathcal{N}}. \end{aligned}$$

Further, from the centrality condition $\Lambda(X(\mu))\Lambda(S(\mu)) = \mu I_n$, we can observe that the i^{th} largest eigenvalue of $X(\mu)$ and the i^{th} smallest eigenvalue of $S(\mu)$ have the same eigenvector, which implies $\lambda_{[i]}(X(\mu))\lambda_{[n-i+1]}(S(\mu)) = \mu$. Hence, we can derive

$$\begin{aligned} \lambda_{[i]}(X(\mu)) &\ge \frac{\sigma}{n}, \quad i = 1, \dots, n_{\mathcal{B}}, \\ \lambda_{[i]}(S(\mu)) &\ge \frac{\sigma}{n}, \quad i = 1, \dots, n_{\mathcal{N}}. \end{aligned}$$

It follows from Lemmas 4 and 5 and $\operatorname{tr}(X) \le \sqrt{n}\|X\|$ that

$$\begin{aligned} \frac{1}{\sqrt{n}}\Big(\lambda_{[n-n_{\mathcal{N}}-n_{\mathcal{T}}+1]}(X(\mu)) + \ldots + \lambda_{[n]}(X(\mu))\Big) &\le \Big\|Q_{\mathcal{T}\cup\mathcal{N}}^T X(\mu) Q_{\mathcal{T}\cup\mathcal{N}}\Big\| \le c(n\mu)^{\gamma}, \\ \frac{1}{\sqrt{n}}\Big(\lambda_{[n-n_{\mathcal{B}}-n_{\mathcal{T}}+1]}(S(\mu)) + \ldots + \lambda_{[n]}(S(\mu))\Big) &\le \Big\|Q_{\mathcal{B}\cup\mathcal{T}}^T S(\mu) Q_{\mathcal{B}\cup\mathcal{T}}\Big\| \le c(n\mu)^{\gamma}, \end{aligned}$$

which by the centrality condition gives (14).

By (14), if $n_{\mathcal{T}} > 0$, there exist $n_{\mathcal{T}}$ eigenvalues of $X(\mu)$ and $n_{\mathcal{T}}$ eigenvalues of $S(\mu)$ which stay within the interval $\left[\frac{\mu}{c\sqrt{n}(n\mu)^{\gamma}},\ c\sqrt{n}(n\mu)^{\gamma}\right]$, and thus both converge to 0 as $\mu \to 0$. Then it holds that

$$c\sqrt{n}(n\mu)^{\gamma} \ge \frac{\mu}{c\sqrt{n}(n\mu)^{\gamma}} \quad \Rightarrow \quad c^2 n^2 \ge (n\mu)^{1-2\gamma}, \quad \forall\ 0 < \mu \le \frac{1}{n},$$

which implies $c \geq \frac{1}{n}$ and $\gamma \leq \frac{1}{2}$. The lower bound $\gamma \geq \frac{1}{2^{n-1}}$ follows from Lemma 11 in Appendix B. This completes the proof.

□

From Theorem 2 one can observe that associated with a central solution $(X(\mu), y(\mu), S(\mu))$, in general, there exist three sets of eigenvectors $q_i(\mu)$ for which

- $\lambda_i(X(\mu))$ converges to a positive value and $\lambda_i(S(\mu))$ converges to 0;
- $\lambda_i(S(\mu))$ converges to a positive value and $\lambda_i(X(\mu))$ converges to 0;
- both $\lambda_i(X(\mu))$ and $\lambda_i(S(\mu))$ converge to 0,

where $\lambda_i(X(\mu))$ and $\lambda_i(S(\mu))$ correspond to the eigenvector $q_i(\mu)$. As $\mu \to 0$, these sets of eigenvectors converge to an orthonormal bases for $\mathcal{B}$, $\mathcal{N}$, and $\mathcal{T}$, see Section 3.3 in [4]. Note that the eigenvectors converge to some orthonormal bases, but not necessarily to Q chosen in Section 2. The following theorem specifies an upper bound for μ which allows the identification of the above sets of eigenvectors.

Theorem 3. *If μ satisfies*

$$\mu < \min\left\{\frac{1}{n}\left(\frac{\sigma}{cn^{\frac{3}{2}}}\right)^{\frac{1}{\gamma}}, \frac{\sigma^2}{n^2}, \frac{1}{n}\right\}, \tag{15}$$

then we can identify the sets of eigenvectors converging to an orthonormal bases for $\mathcal{B}$, $\mathcal{N}$, and $\mathcal{T}$.

Proof. From the inequalities (12) and (13), we can deduce that the $n_\mathcal{B}$ largest eigenvalues of $X(\mu)$ stay positive while the $n_\mathcal{B}$ smallest eigenvalues of $S(\mu)$ will converge to 0. Similarly, the $n_\mathcal{N}$ largest eigenvalues of $S(\mu)$ will remain positive while the last $n_\mathcal{N}$ eigenvalues of $X(\mu)$ converge to 0 as $\mu \to 0$. The inequalities (14) also hint that, if $n_\mathcal{T} > 0$, there should exist a set of $n_\mathcal{T}$ eigenvalues of $X(\mu)$ and $S(\mu)$ which stay within the interval $\left[\frac{\mu}{c\sqrt{n}(n\mu)^\gamma}, c\sqrt{n}(n\mu)^\gamma\right]$. Recall that the i^{th} largest eigenvalue of $X(\mu)$ and the i^{th} smallest eigenvalue of $S(\mu)$ have the same eigenvector. Thus, we can identify the sets of eigenvectors which converge to an orthonormal bases for $\mathcal{B}$, $\mathcal{N}$, and $\mathcal{T}$ if

$$\frac{n\mu}{\sigma} < \frac{\mu}{c\sqrt{n}(n\mu)^\gamma}, \qquad c\sqrt{n}(n\mu)^\gamma < \frac{\sigma}{n}, \qquad \frac{n\mu}{\sigma} < \frac{\sigma}{n}, \tag{16}$$

which, by $\frac{\mu}{c\sqrt{n}(n\mu)^\gamma} \leq c\sqrt{n}(n\mu)^\gamma$, is equivalent to

$$\mu < \frac{1}{n}\left(\frac{\sigma}{cn^{\frac{3}{2}}}\right)^{\frac{1}{\gamma}}. \tag{17}$$

Furthermore, in case that $\mathcal{T} = \{0\}$, μ needs to satisfy

$$\mu \leq \frac{\sigma^2}{n^2}.$$

Finally, $n\mu \leq 1$ must hold as well in order to retain the validity of the bounds in (9). This completes the proof. □

Remark 4. *In reality, we do not know in advance if the strict complementarity condition holds for a given instance of SDO. Note that* (16) *and* (17) *imply that if $n_\mathcal{T} > 0$, then we have*

$$\frac{1}{n}\left(\frac{\sigma}{cn^{\frac{3}{2}}}\right)^{\frac{1}{\gamma}} \leq \frac{\sigma^2}{n^2}.$$

If $n_{\mathcal{T}} = 0$, then we can make improvement on the bound (15)*. In fact, the bounds in* (14) *may provide no further information compared to* (12) *and* (13) *for small values of μ. Hence, in order to identify the eigenvectors converging to an orthonormal bases for $\mathcal{B}$ and $\mathcal{N}$ it is enough to have*

$$\frac{n\mu}{\sigma} < \frac{\sigma}{n},$$

which reduces the bound (15) *to $\mu < \frac{\sigma^2}{n^2}$. This bound matches the one for LO, see Section 3.3.3 in [24].*

4 On the identification of the optimal partition in a neighborhood of the central path

Thus far, we assumed that the solution given by IPMs is exactly on the central path. In reality, however, path-following IPMs operate in a specified vicinity of the central path by computing approximate solutions of (1).

Consider a solution $(X, y, S) \in \mathrm{ri}(\mathcal{P}\times\mathcal{D})$ given by a primal-dual path-following IPM, where $X := M\Lambda(X)M^T$ and $S := P\Lambda(S)P^T$ are eigenvalue decomposition of X and S, respectively, and M and P serve as orthogonal matrices. In contrast to the result of Theorem 3, the sets of eigenvectors converging to an orthonormal bases for $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{T}$ are not identical for X and S. The reason lies in the fact that X and S do not necessarily commute. For instance, consider the Nesterov-Todd scaling method, where X and S are projected onto the same point V defined as

$$V := D^{-\frac{1}{2}}XD^{-\frac{1}{2}} = D^{\frac{1}{2}}SD^{\frac{1}{2}},$$

which implies $X = DSD$, where $D \succ 0$ denotes the scaling matrix, see [18] for the definition of D. Then the spectral decomposition of S yields

$$X = DP\Lambda(S)P^TD. \tag{18}$$

Since $\Lambda(X)$ and $\Lambda(S)$ have nonzero diagonal entries, we may assume that $\Lambda(S) =: \Sigma^{\frac{1}{2}}\Lambda(X)\Sigma^{\frac{1}{2}}$ where $\Sigma^{\frac{1}{2}}$ is a positive definite diagonal matrix. Hence, from (18) we get

$$\Sigma^{-\frac{1}{2}}P^TD^{-1}XD^{-1}P\Sigma^{-\frac{1}{2}} = \Lambda(X).$$

Note that $D^{-1}P\Sigma^{-\frac{1}{2}}$ is an $n \times n$ invertible matrix but not necessarily equal to the orthogonal matrix M. Therefore, there exists an invertible matrix $N \in \mathbb{R}^{n\times n}$ so that

$$M = D^{-1}P\Sigma^{-\frac{1}{2}}N,$$

implying that X and S do not necessarily share the same eigenvector basis.

The proximity of (X, y, S) to the central path can be measured (see e.g., Section 6.4 in [4]) by

$$\kappa(XS) := \frac{\lambda_{\max}(XS)}{\lambda_{\min}(XS)}, \quad (X, y, S) \in \mathrm{ri}(\mathcal{P} \times \mathcal{D}). \tag{19}$$

Notice that XS has the same eigenvalues as $X^{\frac{1}{2}}SX^{\frac{1}{2}}$, i.e., XS has real positive eigenvalues even though it is not necessarily symmetric. Further, it follows from (19) that $\kappa(XS) \geq 1$, and the equality holds only when (X, y, S) is on the central path. The neighborhood of the central path is defined by

$$\mathcal{N}_\kappa(\tau) := \Big\{(X, y, S) \in \mathrm{ri}(\mathcal{P} \times \mathcal{D}) \mid \kappa(XS) \leq \tau\Big\}, \tag{20}$$

where $\tau > 1$. There exists $\tau_2 \geq 1 \geq \tau_1 > 0$ so that $\tau_2 := \tau\tau_1$, and then for $(X, y, S) \in \mathcal{N}_\kappa(\tau)$ we have

$$\tau_1\lambda_{\min}(XS) \leq \lambda_{[i]}(XS) \leq \tau_2\lambda_{\min}(XS), \quad i = 1, \dots, n. \tag{21}$$

Here, we use the application of Weyl theorem[5] in [15] to provide an upper bound for $\lambda_{\min}(XS)$.

Lemma 6 (Corollary 2.3 in [15])**.** *Let X and S be two $n\times n$ symmetric positive semidefinite matrices. Then for $k \leq \min\{\text{rank}(X),\ \text{rank}(S)\}$ we have*

$$\min_{1\leq i\leq k}\{\lambda_{[i]}(X)\lambda_{[k-i+1]}(S)\} \geq \lambda_{[k]}(XS) \geq \max_{k\leq i\leq n}\{\lambda_{[i]}(X)\lambda_{[n+k-i]}(S)\}. \tag{22}$$

Lemma 7. *Let $(X,y,S)\in\mathcal{N}_\kappa(\tau)$. Then we have*

$$\lambda_{[i]}(X)\lambda_{[n-i+1]}(S) \geq \lambda_{\min}(XS), \quad i=1,\dots,n. \tag{23}$$

Proof. The proof is straightforward from the first inequality in (22) and the positive definiteness of X and S. In fact, for the special case $k=n$ there holds that

$$\min\Big\{\lambda_{[1]}(X)\lambda_{[n]}(S),\ \lambda_{[2]}(X)\lambda_{[n-1]}(S),\dots,\lambda_{[n]}(X)\lambda_{[1]}(S)\Big\} \geq \lambda_{\min}(XS),$$

which completes the proof. □

The following theorem generalizes the bounds derived in Theorem 2 to an approximate solution $(X,y,S)\in\mathcal{N}_\kappa(\tau)$.

Theorem 4. *Let $(X,y,S)\in\mathcal{N}_\kappa(\tau)$, and $\mu := \frac{X\bullet S}{n}$. Then it holds that*

1. *For $i=1,\dots,n_{\mathcal{B}}$ we have*

$$\lambda_{[n-i+1]}(S) \leq \frac{n\mu}{\sigma}, \qquad \lambda_{[i]}(X) \geq \frac{\sigma}{n\tau}.$$

2. *For $i=1,\dots,n_{\mathcal{N}}$ we have*

$$\lambda_{[i]}(S) \geq \frac{\sigma}{n\tau}, \qquad \lambda_{[n-i+1]}(X) \leq \frac{n\mu}{\sigma}.$$

Furthermore, we have

$$\begin{aligned}
\lambda_{[n-i+1]}(X) &\leq c\sqrt{n}(n\mu)^\gamma, \qquad \lambda_{[i]}(S) \geq \frac{\mu}{c\sqrt{n}\tau(n\mu)^\gamma}, \qquad i=1,\dots,n_{\mathcal{N}}+n_{\mathcal{T}},\\
\lambda_{[n-i+1]}(S) &\leq c\sqrt{n}(n\mu)^\gamma, \qquad \lambda_{[i]}(X) \geq \frac{\mu}{c\sqrt{n}\tau(n\mu)^\gamma}, \qquad i=1,\dots,n_{\mathcal{B}}+n_{\mathcal{T}}.
\end{aligned}$$

If $n_{\mathcal{T}}>0$, then we have

$$\frac{1}{2^{n-1}} \leq \gamma \leq \frac{1}{2}.$$

If μ satisfies

$$\mu < \min\left\{\frac{1}{n}\left(\frac{\sigma}{cn^{\frac{3}{2}}\tau}\right)^{\frac{1}{\gamma}},\ \frac{\sigma^2}{n^2\tau},\ \frac{1}{n}\right\}, \tag{24}$$

then we can identify the sets of eigenvectors converging to an orthonormal bases for $\mathcal{B}$, $\mathcal{N}$, and $\mathcal{T}$.

Proof. The proof technique can be traced back to Theorem 2 fairly easily. Let $(\bar{X},\bar{y},\bar{S})\in\mathcal{P}^*\times\mathcal{D}^*$ which satisfies (11), $(X,y,S)\in\mathcal{N}_\kappa(\tau)$, and $(\hat{X},\hat{S})$ denote the orthogonal transformation of (X,S) with respect to Q. Then it follows from the orthogonality between $(X-\bar{X})$ and $(S-\bar{S})$ that

$$X\bullet\bar{S}+\bar{X}\bullet S = \hat{X}_{\mathcal{N}}\bullet U_{\bar{S}} + U_{\bar{X}}\bullet\hat{S}_{\mathcal{B}} = X\bullet S,$$

[5] See Theorem 4.3.7 in [12].

where $\hat{S}_{\mathcal{B}} = Q_{\mathcal{B}}^T S Q_{\mathcal{B}}$ and $\hat{X}_{\mathcal{N}} = Q_{\mathcal{N}}^T X Q_{\mathcal{N}}$. Using the inequality $\lambda_{\min}(U_{\bar{S}}) \operatorname{tr}(\hat{X}_{\mathcal{N}}) \leq \hat{X}_{\mathcal{N}} \bullet U_{\bar{S}}$ and the positive definiteness of X and S we have

$$\begin{aligned} \lambda_{\min}(U_{\bar{X}}) \operatorname{tr}(\hat{S}_{\mathcal{B}}) &\leq X \bullet S \quad \Rightarrow \quad \operatorname{tr}(\hat{S}_{\mathcal{B}}) &\leq \frac{n\mu}{\sigma}, \\ \lambda_{\min}(U_{\bar{S}}) \operatorname{tr}(\hat{X}_{\mathcal{N}}) &\leq X \bullet S \quad \Rightarrow \quad \operatorname{tr}(\hat{X}_{\mathcal{N}}) &\leq \frac{n\mu}{\sigma}, \end{aligned}$$

where the latter inequalities follow from (11). Now, Lemma 5 can be applied to get

$$\begin{aligned} \lambda_{[n-n_{\mathcal{B}}+1]}(S) + \ldots + \lambda_{[n]}(S) &\leq \operatorname{tr}(\hat{S}_{\mathcal{B}}) &\leq \frac{n\mu}{\sigma}, \\ \lambda_{[n-n_{\mathcal{N}}+1]}(X) + \ldots + \lambda_{[n]}(X) &\leq \operatorname{tr}(\hat{X}_{\mathcal{N}}) &\leq \frac{n\mu}{\sigma}, \end{aligned}$$

which by $X, S \succ 0$ implies

$$\begin{aligned} \lambda_{[n-i+1]}(S) &\leq \frac{n\mu}{\sigma}, \qquad i = 1, \ldots, n_{\mathcal{B}}, \\ \lambda_{[n-i+1]}(X) &\leq \frac{n\mu}{\sigma}, \qquad i = 1, \ldots, n_{\mathcal{N}}. \end{aligned} \tag{25}$$

Recall from (21) that

$$n\mu = X \bullet S \leq n\tau_2 \lambda_{\min}(XS),$$

which yields

$$\frac{\lambda_{\min}(XS)}{\mu} \geq \frac{1}{\tau_2} \geq \frac{1}{\tau}, \tag{26}$$

where $\tau_2 = \tau\tau_1$ and $\tau_1 \leq 1$. Then (23) and (26) can be applied to (25) to derive lower bounds for the eigenvalues of X and S:

$$\begin{aligned} \lambda_{[i]}(X) &\geq \frac{\lambda_{\min}(XS)}{\lambda_{[n-i+1]}(S)} \geq \frac{\sigma\lambda_{\min}(XS)}{n\mu} \geq \frac{\sigma}{n\tau}, \qquad i = 1, \ldots, n_{\mathcal{B}}, \\ \lambda_{[i]}(S) &\geq \frac{\lambda_{\min}(XS)}{\lambda_{[n-i+1]}(X)} \geq \frac{\sigma\lambda_{\min}(XS)}{n\mu} \geq \frac{\sigma}{n\tau}, \qquad i = 1, \ldots, n_{\mathcal{N}}. \end{aligned}$$

For the $\mathcal{T}$ subspace we should note that

$$\left\{(X, y, S) \in \mathcal{N}_{\kappa}(\tau) \mid X \bullet S \leq 1\right\}$$

is a bounded set by the interior point condition and the linear independence of A^i for $i = 1, \ldots, m$, see e.g. Lemma 3.1 in [4]. Furthermore, the amount of constraint violation with respect to the LMI system (10) for (X, y, S) is equal to $n\mu$. Hence, the result of Lemma 10 is still valid, i.e., for $0 < \mu \leq \frac{1}{n}$ there exists $(\tilde{X}, \tilde{y}, \tilde{S}) \in \mathcal{P}^* \times \mathcal{D}^*$ so that

$$\left\|X - \tilde{X}\right\| \leq c(n\mu)^{\gamma}, \quad \left\|S - \tilde{S}\right\| \leq c(n\mu)^{\gamma},$$

where c and γ are defined as in Lemma 3. Analogous to the proof of Theorem 2, we can observe, using the orthogonal transformation Q, that

$$\begin{aligned} \left\|Q_{\mathcal{T}\cup\mathcal{N}}^T X Q_{\mathcal{T}\cup\mathcal{N}}\right\| &= \left\|\begin{bmatrix} \hat{X}_{\mathcal{T}} & \hat{X}_{\mathcal{T}\mathcal{N}} \\ \hat{X}_{\mathcal{N}\mathcal{T}} & \hat{X}_{\mathcal{N}} \end{bmatrix}\right\| &\leq \left\|X - \tilde{X}\right\| &\leq c(n\mu)^{\gamma}, \\ \left\|Q_{\mathcal{B}\cup\mathcal{T}}^T S Q_{\mathcal{B}\cup\mathcal{T}}\right\| &= \left\|\begin{bmatrix} \hat{S}_{\mathcal{B}} & \hat{S}_{\mathcal{B}\mathcal{T}} \\ \hat{S}_{\mathcal{T}\mathcal{B}} & \hat{S}_{\mathcal{T}} \end{bmatrix}\right\| &\leq \left\|S - \tilde{S}\right\| &\leq c(n\mu)^{\gamma}. \end{aligned} \tag{27}$$

Then it follows from Lemma 5 and (27) that

$$\begin{aligned} \lambda_{[n-n_{\mathcal{N}}-n_{\mathcal{T}}+1]}(X) + \ldots + \lambda_{[n]}(X) &\le c\sqrt{n}(n\mu)^{\gamma}, \\ \lambda_{[n-n_{\mathcal{B}}-n_{\mathcal{T}}+1]}(S) + \ldots + \lambda_{[n]}(S) &\le c\sqrt{n}(n\mu)^{\gamma}, \end{aligned}$$

and consequently,

$$\begin{aligned} \lambda_{[n-i+1]}(X) &\le c\sqrt{n}(n\mu)^{\gamma}, \qquad i = 1, \ldots, n_{\mathcal{N}} + n_{\mathcal{T}}, \\ \lambda_{[n-i+1]}(S) &\le c\sqrt{n}(n\mu)^{\gamma}, \qquad i = 1, \ldots, n_{\mathcal{B}} + n_{\mathcal{T}}. \end{aligned}$$

Using the bounds in (23) and (26) we can derive

$$\begin{aligned} \lambda_{[i]}(X) &\ge \frac{\lambda_{\min}(XS)}{\lambda_{[n-i+1]}(S)} \ge \frac{\lambda_{\min}(XS)}{c\sqrt{n}(n\mu)^{\gamma}} \ge \frac{\mu}{c\sqrt{n}\tau(n\mu)^{\gamma}}, \qquad i = 1, \ldots, n_{\mathcal{B}} + n_{\mathcal{T}}, \\ \lambda_{[i]}(S) &\ge \frac{\lambda_{\min}(XS)}{\lambda_{[n-i+1]}(X)} \ge \frac{\lambda_{\min}(XS)}{c\sqrt{n}(n\mu)^{\gamma}} \ge \frac{\mu}{c\sqrt{n}\tau(n\mu)^{\gamma}}, \qquad i = 1, \ldots, n_{\mathcal{N}} + n_{\mathcal{T}}. \end{aligned}$$

In the sequel, using the same argument as in Theorem 3, we can identify the sets of eigenvectors which converge to an orthonormal bases for $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{T}$ if

$$\frac{n\mu}{\sigma} < \frac{\mu}{c\sqrt{n}\tau(n\mu)^{\gamma}}, \qquad c\sqrt{n}(n\mu)^{\gamma} < \frac{\sigma}{n\tau}. \tag{28}$$

Considering the case $\mathcal{T} = \{0\}$, we can represent (28) as

$$\mu < \min\left\{\frac{1}{n}\left(\frac{\sigma}{cn^{\frac{3}{2}}\tau}\right)^{\frac{1}{\gamma}}, \ \frac{\sigma^2}{n^2\tau}\right\}.$$

Including the condition $\mu \le \frac{1}{n}$ gives the result as desired. Further, if $n_{\mathcal{T}} > 0$, from $\dfrac{\mu}{c\sqrt{n}\tau(n\mu)^{\gamma}} \le c\sqrt{n}(n\mu)^{\gamma}$ we get

$$c^2 n^2 \tau \ge (n\mu)^{1-2\gamma}, \quad \forall\ 0 < \mu \le \frac{1}{n},$$

which implies

$$\frac{1}{2^{n-1}} \le \gamma \le \frac{1}{2}.$$

This completes the proof.

□

Corollary 1. *Let* $(X^0, y^0, S^0) \in \mathcal{N}_{\kappa}(\tau)$, $\mu^0 := \frac{X^0 \bullet S^0}{n}$, *and* $\log(.)$ *denote the natural logarithm. Then the Dikin-type primal-dual affine scaling method with steplength* $\alpha = \frac{1}{\tau\sqrt{n}}$ *and the neighborhood* (20) *(see Section 6.6 in [4]) needs at most*

$$\left\lceil \tau n \log\left(\mu^0\left(\min\left\{\frac{1}{n}\left(\frac{\sigma}{cn^{\frac{3}{2}}\tau}\right)^{\frac{1}{\gamma}}, \ \frac{\sigma^2}{n^2\tau}, \ \frac{1}{n}\right\}\right)^{-1}\right)\right\rceil$$

iterations to get an $(X, y, S) \in \mathcal{N}_{\kappa}(\tau)$ *which allows to identify the sets of eigenvectors converging to an orthonormal bases for* $\mathcal{B}$, $\mathcal{N}$, *and* $\mathcal{T}$.

Proof. The proof easily follows from the iteration complexity result for the Dikin-type primal-dual affine scaling method with steplength $\alpha = \frac{1}{\tau\sqrt{n}}$ [4]. Then the complementarity gap drops below a certain threshold ϵ_c after

$$\left\lceil \tau n \log\left(\frac{n\mu^0}{\epsilon_c}\right)\right\rceil$$

iterations. The result follows if we replace ϵ_c by the right hand side of (24) multiplied by n. □

5 Concluding remarks

In this paper, we considered the identification of the optimal partition for SDO where strict complementarity may fail. Using a condition number and the error bound result for LMIs, we derived bounds for the magnitude of the eigenvalues of a primal-dual solution on, or in a neighborhood of the central path. We then used the bounds to identify the sets of eigenvectors which converge to an orthonormal bases for $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{T}$. An iteration complexity bound was provided which states that the Dikin-type primal-dual affine scaling algorithm needs at most

$$\left\lceil \tau n \log\left(\mu^0 \left(\min\left\{ \frac{1}{n}\left(\frac{\sigma}{cn^{\frac{3}{2}}\tau}\right)^{\frac{1}{\gamma}},\ \frac{\sigma^2}{n^2\tau},\ \frac{1}{n} \right\}\right)^{-1}\right)\right\rceil$$

iterations to identify the above sets of eigenvectors.

Acknowledgements

This work is supported by the Air force Office of Scientific Research (AFOSR) Grant # FA9550-15-1-0222.

References

[1] F. Alizadeh. *Combinatorial Optimization with Interior Point Methods and Semidefinite Matrices*. PhD thesis, University of Minnesota, 1991.

[2] J. F. Bonnans and H. Ramírez C. Perturbation analysis of second-order cone programming problems. *Mathematical Programming*, 104(2):205–227, 2005.

[3] Y.-L. Cheung. *Preprocessing and Reduction for Semidefinite Programming via Facial Reduction: Theory and Practice*. PhD thesis, Department of Combinatorics & Optimization, University of Waterloo, 2013.

[4] E. de Klerk. *Aspects of Semidefinite Programming: Interior Point Algorithms and Selected Applications*, volume 65. Springer Science & Business Media, 2006.

[5] E. de Klerk, C. Roos, and T. Terlaky. Initialization in semidefinite programming via a self-dual skew-symmetric embedding. *Operations Research Letters*, 20(5):213–221, 1997.

[6] A. M. Geoffrion. Generalized Benders decomposition. *Journal of Optimization Theory and Applications*, 10(4):237–260, 1972.

[7] D. Goldfarb and K. Scheinberg. Interior point trajectories in semidefinite programming. *SIAM Journal on Optimization*, 8(4):871–886, 1998.

[8] D. Goldfarb and K. Scheinberg. On parametric semidefinite programming. *Applied Numerical Mathematics*, 29(3):361 – 377, 1999.

[9] O. Güler, D. den Hertog, C. Roos, T. Terlaky, and T. Tsuchiya. Degeneracy in interior point methods for linear programming: a survey. *Annals of Operations Research*, 46(1):107–138, 1993.

[10] M. Halická. Analyticity of the central path at the boundary point in semidefinite programming. *European Journal of Operational Research*, 143(2):311–324, 2002.

[11] M. Halická, E. de Klerk, and C. Roos. On the convergence of the central path in semidefinite optimization. *SIAM Journal on Optimization*, 12(4):1090–1099, 2002.

[12] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 2012.

[13] T. Illés, J. Peng, C. Roos, and T. Terlaky. A strongly polynomial rounding procedure yielding a maximally complementary solution for $P_*(\kappa)$ linear complementarity problems. *SIAM Journal on Optimization*, 11(2):320–340, 2000.

[14] F. Jarre. Convex analysis on symmetric matrices. In H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors, *Handbook of Semidefinite Programming*, pages 13–27. Springer, 2000.

[15] L.-Z. Lu and C. Pearce. Some new bounds for singular values and eigenvalues of matrix products. *Annals of Operations Research*, 98(1):141–148, 2000.

[16] Z.-Q. Luo, J. F. Sturm, and S. Zhang. Superlinear convergence of a symmetric primal-dual path following algorithm for semidefinite programming. *SIAM Journal on Optimization*, 8(1):59–81, 1998.

[17] Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*, volume 13. Society for Industrial and Applied Mathematics, 1994.

[18] Y. Nesterov and M. J. Todd. Self-scaled barriers and interior-point methods for convex programming. *Mathematics of Operations Research*, 22(1):1–42, 1997.

[19] J. Nocedal and S. Wright. *Numerical Optimization.* Springer Science & Business Media, second edition, 2006.

[20] J.-S. Pang. Error bounds in mathematical programming. *Mathematical Programming*, 79(1):299–332, 1997.

[21] G. Pataki. Strong duality in conic linear programming: Facial reduction and extended duals. In D. Bailey, H. Bauschke, P. Borwein, F. Garvan, M. Théra, J. Vanderwerff, and H. Wolkowicz, editors, *Computational and Analytical Mathematics*, volume 50 of Springer Proceedings in Mathematics & Statistics, pages 613–634. Springer, 2013.

[22] J. Peña and V. Roshchina. A complementarity partition theorem for multifold conic systems. *Mathematical Programming*, 142(1):579–589, 2013.

[23] M. V. Ramana. *An algorithmic Analysis of Multiquadratic and Semidefinite Programming Problems.* PhD thesis, Johns Hopkins University, 1993.

[24] C. Roos, T. Terlaky, and J.-P. Vial. *Interior Point Methods for Linear Optimization.* Springer Science & Business Media, 2005.

[25] G. W. Stewart and J.-g. Sun. *Matrix Perturbation Theory.* Academic Press, 1990.

[26] J. F. Sturm. Error bounds for linear matrix inequalities. *SIAM Journal on Optimization*, 10(4):1228–1248, 2000.

[27] T. Terlaky and Z. Wang. On the identification of the optimal partition of second order cone optimization problems. *SIAM Journal on Optimization*, 24(1):385–414, 2014.

[28] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, 1996.

[29] Y. Ye. On the finite convergence of interior-point algorithms for linear programming. *Mathematical Programming*, 57(1):325–335, 1992.

[30] E. Yildirim. Unifying optimal partition approach to sensitivity analysis in conic optimization. *Journal of Optimization Theory and Applications*, 122(2):405–423, 2004.

A A lower bound for σ

In this section, we derive a lower bound for the condition number σ defined in (6). To do so, we resort to a technical lemma in [23].
An integral polynomial map $f : \mathbb{R}^s \to \mathbb{R}^t$ is defined as a map consisting of polynomial functions p_i of degree d_i with integer coefficients. We consider a solution set $V(f)$ defined as

$$V(f) := \{x \mid p_i(x)\ \Delta_i\ 0, \quad \forall\ i\},$$

where Δ_i stands for one of the relations $\{>, =, \geq\}$. Depending on the polynomial map f, the solution set $V(f)$ could be connected or disconnected. For this polynomial map L_f denotes the binary length of the largest absolute value of the coefficients of the polynomials, where the binary length of an integer n is defined as

$$l(n) := 1 + \lceil \log_2(|n| + 1) \rceil,$$

in which $\log_2(.)$ stands for the logarithm to the base 2.

The next lemma shows that there exists a sphere $B(0, r)$ which circumscribes some solutions from every connected component of $V(f)$.

Lemma 8 (Lemma 3.1 in [23])**.** *Suppose that the polynomials in the polynomial map f have maximum degree d, i.e., $d := \max_i\{d_i\}$ with $d \geq 2$. Then every connected component of $V(f)$ intersects the sphere $\{x \mid \|x\| \leq r\}$, where $\log_2(r) = L_f(td)^s$.*

Lemma 9. *Let the SDO problems (P) and (D) be given by integer data, L denote the binary length of the largest absolute value of the entries in b, C, and A^i for $i = 1, \ldots, m$, and $\|.\|$ be the Frobenius norm. Then, for the condition number σ we have*

$$\sigma \geq \min\left\{\frac{1}{r_{\mathcal{P}^*}\sum_{i=1}^m \|A^i\|}, \frac{1}{r_{\mathcal{D}^*}}\right\}, \tag{29}$$

where

$$\log_2(r_{\mathcal{P}^*}) = (L+2)\Big(\bar{d}_p(3n_{\mathcal{B}}^2 + n_{\mathcal{N}}^2 + n_{\mathcal{B}}n_{\mathcal{N}} + n_{\mathcal{B}} + n_{\mathcal{N}} + n^2 + m)\Big)^{2n_{\mathcal{B}}^2 + n_{\mathcal{N}}^2 + n(n_{\mathcal{B}} + n_{\mathcal{N}}) + 2m},$$
$$\log_2(r_{\mathcal{D}^*}) = (L+2)\Big(\bar{d}_d(n_{\mathcal{B}}^2 + 3n_{\mathcal{N}}^2 + n_{\mathcal{B}}n_{\mathcal{N}} + n_{\mathcal{B}} + n_{\mathcal{N}} + 2n^2 + 2m)\Big)^{n_{\mathcal{B}}^2 + 2n_{\mathcal{N}}^2 + n^2 + n(n_{\mathcal{B}} + n_{\mathcal{N}}) + m},$$
$$\bar{d}_p = \bar{d}_d = \max\{n_{\mathcal{B}}, n_{\mathcal{N}}, 3\}.$$

Proof. Recall from (4) and (5) that

$$\sigma_{\mathcal{B}} \geq \lambda_{\min}(Q_{\mathcal{B}}^T X Q_{\mathcal{B}}), \quad \sigma_{\mathcal{N}} \geq \lambda_{\min}(Q_{\mathcal{N}}^T X Q_{\mathcal{N}}), \quad \forall (X, y, S) \in \mathcal{P}^* \times \mathcal{D}^*,$$

which motivates us to find a solution in the relative interior of the optimal set. We apply the definition of the analytic center of the optimal set to find a solution in the relative interior of the optimal set, and we then derive a lower bound for its minimum eigenvalue. It should be noted that Ramana [23] used this definition to compute a lower bound for the volume of a sphere inscribed in the feasible set of a so called strict semidefinite feasibility problem.

Throughout the proof, we can assume that $n_{\mathcal{B}}, n_{\mathcal{N}} > 0$. By Theorem 1, any primal-dual optimal pair is a solution to the following LMI system

$$\begin{cases} A^i \bullet Q_{\mathcal{B}} U_X Q_{\mathcal{B}}^T & = b_i, \quad i = 1, \ldots, m, \\ C - \displaystyle\sum_{i=1}^m y_i A^i & = Q_{\mathcal{N}} U_S Q_{\mathcal{N}}^T, \\ U_X, U_S \succeq 0, & \end{cases} \tag{30}$$

where $U_X \in \mathbb{S}_+^{n_\mathcal{B}}$ and $U_S \in \mathbb{S}_+^{n_\mathcal{N}}$ as defined in Theorem 1, and $Q_\mathcal{B}$ and $Q_\mathcal{N}$ are assumed to be known. Therefore, since $n_\mathcal{B}, n_\mathcal{N} > 0$, we obtain the set of maximally complementary optimal solutions if we add the constraints $U_X, U_S \succ 0$ to (30), i.e.,

$$\begin{cases} A^i \bullet Q_\mathcal{B} U_X Q_\mathcal{B}^T & = b_i, \quad i = 1, \dots, m, \\ C - \displaystyle\sum_{i=1}^m y_i A^i & = Q_\mathcal{N} U_S Q_\mathcal{N}^T, \\ U_X, U_S \succ 0. \end{cases} \tag{31}$$

Then for a given orthonormal basis $Q_\mathcal{B}$, the analytic center of the primal optimal set can be computed by solving

$$\begin{aligned} \max \quad & \log(\det(U_{X^a})) \\ \text{s.t.} \quad & A^i \bullet Q_\mathcal{B} U_{X^a} Q_\mathcal{B}^T = b_i, \quad i = 1, \dots, m, \\ & U_{X^a} \succ 0. \end{aligned} \tag{32}$$

Problem (32) is convex with a strictly concave objective function over the cone of positive definite matrices, which by $n_\mathcal{B} > 0$ induces the existence of a unique optimal solution for (32). Further, there exists a vector of Lagrange multipliers $u \in \mathbb{R}^m$ so that the following system of optimality conditions has a solution:

$$\begin{cases} U_{X^a}^{-1} - \sum_{i=1}^m u_i Q_\mathcal{B}^T A^i Q_\mathcal{B} & = 0, \\ A^i \bullet Q_\mathcal{B} U_{X^a} Q_\mathcal{B}^T & = b_i, \quad i = 1, \dots, m, \\ U_{X^a} \succ 0. \end{cases} \tag{33}$$

For any solution (U_{X^a}, u) of (33), which is unique in terms of U_{X^a} but not necessarily in terms of u, $X^a := Q_\mathcal{B} U_{X^a} Q_\mathcal{B}^T$ is the analytic center of the primal optimal set. To derive a lower bound for the minimum eigenvalue of X^a, we have from (33) that

$$\begin{aligned} \lambda_{\min}(U_{X^a}) = \frac{1}{\lambda_{\max}\Big(\sum_{i=1}^m u_i Q_\mathcal{B}^T A^i Q_\mathcal{B}\Big)} &\geq \frac{1}{\|\sum_{i=1}^m u_i Q_\mathcal{B}^T A^i Q_\mathcal{B}\|} \\ &\geq \frac{1}{\sum_{i=1}^m |u_i| \|Q_\mathcal{B}^T A^i Q_\mathcal{B}\|} \\ &\geq \frac{1}{\sum_{i=1}^m |u_i| \|A^i\|}, \end{aligned} \tag{34}$$

where we have used the triangle inequality and the fact that $\|Q_\mathcal{B}^T A^i Q_\mathcal{B}\| \leq \|A^i\|$. Note that the bound (34) depends on an upper bound for $|u_i|$ which itself relies on $Q_\mathcal{B}$. In reality, however, $Q_\mathcal{B}$ is not known a priori, since it is determined by solutions in the relative interior of the optimal set. Hence, the idea is to characterize all possible orthonormal bases for $\mathcal{B}$, i.e., to characterize the properties of $\Gamma_\mathcal{B}$, in the optimality conditions (33) to describe the analytic center of the optimal set. Then a direct application of Lemma 8 to the embedded set yields an upper bound for $|u_i|$.

Assume that $Q_\mathcal{B}$ is an unknown orthonormal basis in (32), i.e., $Q_\mathcal{B}$ is still an orthonormal basis for $\mathcal{B}$ but acts as an unknown in (32), which leads to a nonconvex optimization problem in $Q_\mathcal{B}$ and U_{X^a} jointly. Then, problem (32) can equivalently be written, see e.g., Theorem 2.1 in [6], as

$$\max_{Q_\mathcal{B} \in \Gamma_\mathcal{B}} \max_{U_{X^a} \succ 0} \Big\{ \log(\det(U_{X^a})) : A^i \bullet Q_\mathcal{B} U_{X^a} Q_\mathcal{B}^T = b_i, \quad i = 1, \dots, m \Big\}. \tag{35}$$

In this case, any optimal solution $(Q_\mathcal{B}, U_{X^a})$ of (32) is also optimal for (35) and vice versa. This is due to the fact that the optimal solution of the inner maximization problem in (35) is attained. By Lemma 1, Theorem 1 and (31), the set $\Gamma_\mathcal{B}$ is compact, and it is equivalent to the set of all $Q_\mathcal{B}$ with orthonormal columns by which (31) is feasible. Since the unique optimal solution of the inner maximization problem in (35) is attained, and its set of Lagrange multipliers is nonempty, then (33) with $\Gamma_{\mathcal{B}}$ describes the analytic center of the primal optimal set, see Section 4.2 in [6] for a similar argument in the context of the generalized Benders decomposition.

Now, we apply Lemma 8 to the above embedded set. Let

$$\vartheta_p := (U_{X^a}, u, Z_X, U_S, y, Q_{\mathcal{B}}, Q_{\mathcal{N}}),$$

where $Z_X \in \mathbb{R}^{n_{\mathcal{B}} \times n_{\mathcal{B}}}$, and $\mathrm{vec} : \mathbb{R}^{n \times n} \to \mathbb{R}^{n^2}$ be the concatenation of the columns of an $n \times n$ matrix. We then define the integral polynomial map

$$f_p : \mathbb{R}^{n_{\mathcal{B}} \times n_{\mathcal{B}}} \times \mathbb{R}^m \times \mathbb{R}^{n_{\mathcal{B}} \times n_{\mathcal{B}}} \times \mathbb{R}^{n_{\mathcal{N}} \times n_{\mathcal{N}}} \times \mathbb{R}^m \times \mathbb{R}^{n \times n_{\mathcal{B}}} \times \mathbb{R}^{n \times n_{\mathcal{N}}} \to \mathbb{R}^{t_p}$$

as defined below

$$f_p(\vartheta_p) := \begin{bmatrix} \mathrm{vec}\Big(Z_X - \sum_{i=1}^m u_i Q_{\mathcal{B}}^T A^i Q_{\mathcal{B}}\Big) \\ \mathrm{vec}\big(U_{X^a} Z_X - I_{n_{\mathcal{B}}}\big) \\ A^1 \bullet Q_{\mathcal{B}} U_{X^a} Q_{\mathcal{B}}^T - b_1 \\ \vdots \\ A^m \bullet Q_{\mathcal{B}} U_{X^a} Q_{\mathcal{B}}^T - b_m \\ \mathrm{vec}\Big(C - \sum_{i=1}^m y_i A^i - Q_{\mathcal{N}} U_S Q_{\mathcal{N}}^T\Big) \\ \mathrm{vec}\big(Q_{\mathcal{B}}^T Q_{\mathcal{B}} - I_{n_{\mathcal{B}}}\big) \\ \mathrm{vec}\big(Q_{\mathcal{N}}^T Q_{\mathcal{N}} - I_{n_{\mathcal{N}}}\big) \\ \mathrm{vec}\big(Q_{\mathcal{B}}^T Q_{\mathcal{N}}\big) \end{bmatrix}, \tag{36}$$

where $t_p = 3n_{\mathcal{B}}^2 + n_{\mathcal{N}}^2 + n_{\mathcal{B}} n_{\mathcal{N}} + n^2 + m$. Note that the symmetry of Z_X and U_S follows from the symmetry of A^i and C, and the symmetry of U_{X^a} follows from the symmetry of Z_X. Moreover, we define the solution set Ω_p to enforce the positive definiteness of U_{X^a} and U_S as follows

$$\Omega_p := \Big\{\vartheta_p \mid \det(U_{X^a}[i]) > 0, \ \ \det(U_S[j]) > 0, \ \ i = 1, \ldots, n_{\mathcal{B}}, \ j = 1, \ldots, n_{\mathcal{N}}\Big\}, \tag{37}$$

in which $U_{X^a}[i]$ denotes the i^{th} leading principal submatrix of U_{X^a}. Indeed, the strict inequalities in (37) are necessary and sufficient for the positive definiteness of U_{X^a} and U_S. By the interior point assumption, the solution set $V(f_p) \cap \Omega_p$, where $V(f_p) = \big\{\vartheta_p \mid f_p(\vartheta_p) = 0\big\}$, is nonempty but not necessarily a singleton. Then, from every solution $\vartheta_p \in V(f_p) \cap \Omega_p$, we can extract a solution $(U_{X^a}, u, Q_{\mathcal{B}})$ which is the analytic center of the primal optimal set.

The solution set Ω_p is characterized by $n_{\mathcal{B}} + n_{\mathcal{N}}$ integer polynomials of the maximum degree $\max\{n_{\mathcal{B}}, n_{\mathcal{N}}\}$. Since the symmetry of the matrices U_{X^a}, Z_X, and U_S is not presumed for f_p and Ω_p, the coefficients of the polynomial functions are bounded above by twice the largest absolute value of the entries in b, C, and A^i for $i = 1, \ldots, m$. For instance, the coefficients of $\det(U_{X^a}[i])$ are just 1, but $u_i Q_{\mathcal{B}}^T A^i Q_{\mathcal{B}}$ has some polynomial terms with coefficients twice the off-diagonal entries of A^i. Hence, the binary length of the largest absolute value of the coefficients in (36) and (37) is bounded above by $L + l(2) - 1 = L + 2$.

Consequently, by applying Lemma 8 to the set $V(f_p) \cap \Omega_p$, we can conclude that there exists a solution $\vartheta_p \in V(f_p) \cap \Omega_p$ so that $\|\vartheta_p\| \leq r_{\mathcal{P}^*}$, where

$$\begin{aligned} &\log_2(r_{\mathcal{P}^*}) = (L+2)(\bar{t}_p \bar{d}_p)^{\bar{s}_p}, \\ &\bar{d}_p := \max\{n_{\mathcal{B}}, n_{\mathcal{N}}, 3\}, \\ &\bar{t}_p := t_p + n_{\mathcal{B}} + n_{\mathcal{N}} = 3n_{\mathcal{B}}^2 + n_{\mathcal{N}}^2 + n_{\mathcal{B}} n_{\mathcal{N}} + n_{\mathcal{B}} + n_{\mathcal{N}} + n^2 + m, \\ &\bar{s}_p = 2n_{\mathcal{B}}^2 + n_{\mathcal{N}}^2 + n(n_{\mathcal{B}} + n_{\mathcal{N}}) + 2m, \end{aligned}$$

in which $\bar{s}_p$ denotes the total number of variables in the polynomial map f_p, and $\bar{d}_p$ is the maximum degree of the polynomials in f_p and the polynomials defining Ω_p. As a result, there exists u so that $|u_i| \leq \|u\| \leq r_{\mathcal{P}^*}$. Then, using the inequality (34), we get

$$\sigma_{\mathcal{B}} \geq \lambda_{\min}(U_{X^a}) \geq \frac{1}{\sum_{i=1}^m |u_i| \|A^i\|} \geq \frac{1}{r_{\mathcal{P}^*} \sum_{i=1}^m \|A^i\|}.$$

This completes the first part of the proof. In a similar fashion, we can use the same reasoning as in the primal side to derive a lower bound for $\sigma_{\mathcal{N}}$. Notice that for a given orthonormal basis $Q_{\mathcal{N}}$, the analytic center of the dual optimal set can be obtained by solving

$$\begin{aligned} \max \quad & \log(\det(U_{S^a})) \\ \text{s.t.} \quad & \sum_{i=1}^m y_i^a A^i + Q_{\mathcal{N}} U_{S^a} Q_{\mathcal{N}}^T = C, \\ & U_{S^a} \succ 0, \end{aligned} \tag{38}$$

which is a convex optimization problem with strictly concave objective function. The optimality conditions for (38) are given by

$$\begin{cases} U_{S^a}^{-1} - Q_{\mathcal{N}}^T W Q_{\mathcal{N}} & = 0, \\ A^i \bullet W & = 0, \quad i = 1, \dots, m, \\ \sum_{i=1}^m y_i^a A^i + Q_{\mathcal{N}} U_{S^a} Q_{\mathcal{N}}^T & = C, \\ U_{S^a} \succ 0, \end{cases} \tag{39}$$

where W is an $n \times n$ symmetric matrix. Note that the symmetry of A^i induces the symmetry of U_{S^a} but not necessarily the symmetry[6] of W. Then the optimality conditions (39) imply

$$\lambda_{\min}(U_{S^a}) = \frac{1}{\lambda_{\max}(Q_{\mathcal{N}}^T W Q_{\mathcal{N}})} \geq \frac{1}{\|Q_{\mathcal{N}}^T W Q_{\mathcal{N}}\|} \geq \frac{1}{\|W\|}. \tag{40}$$

Let $\vartheta_d := \big(U_{S^a}, y^a, U_X, Z_S, W, Q_{\mathcal{B}}, Q_{\mathcal{N}}\big)$, where $Z_S \in \mathbb{R}^{n_{\mathcal{N}} \times n_{\mathcal{N}}}$, and consider the solution set

$$V(f_d) := \Big\{ \vartheta_d \mid f_d(\vartheta_d) = 0 \Big\},$$

where the integral polynomial map

$$f_d : \mathbb{R}^{n_{\mathcal{N}} \times n_{\mathcal{N}}} \times \mathbb{R}^m \times \mathbb{R}^{n_{\mathcal{B}} \times n_{\mathcal{B}}} \times \mathbb{R}^{n_{\mathcal{N}} \times n_{\mathcal{N}}} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n_{\mathcal{B}}} \times \mathbb{R}^{n \times n_{\mathcal{N}}} \to \mathbb{R}^{t_d}$$

[6]Note that $Q_{\mathcal{N}}^T (W - W^T) Q_{\mathcal{N}} = 0$ does not necessarily imply $W = W^T$. For example, let $W = \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$ and $Q_{\mathcal{N}} = [1,\ 0]^T$. Then $Q_{\mathcal{N}}^T W Q_{\mathcal{N}} = 1 > 0$ even though W is not symmetric.

is defined as

$$f_d(\vartheta_d) := \begin{bmatrix} \mathrm{vec}\big(Z_S - Q_{\mathcal{N}}^T W Q_{\mathcal{N}}\big) \\ \mathrm{vec}\big(U_{S^a} Z_S - I_{n_{\mathcal{N}}}\big) \\ A^1 \bullet W \\ \vdots \\ A^m \bullet W \\ A^1 \bullet Q_{\mathcal{B}} U_X Q_{\mathcal{B}}^T - b_1 \\ \vdots \\ A^m \bullet Q_{\mathcal{B}} U_X Q_{\mathcal{B}}^T - b_m \\ \mathrm{vec}\Big(C - \sum_{i=1}^m y_i^a A^i - Q_{\mathcal{N}} U_{S^a} Q_{\mathcal{N}}^T\Big) \\ \mathrm{vec}\big(W - W^T\big) \\ \mathrm{vec}\big(Q_{\mathcal{B}}^T Q_{\mathcal{B}} - I_{n_{\mathcal{B}}}\big) \\ \mathrm{vec}\big(Q_{\mathcal{N}}^T Q_{\mathcal{N}} - I_{n_{\mathcal{N}}}\big) \\ \mathrm{vec}\big(Q_{\mathcal{B}}^T Q_{\mathcal{N}}\big) \end{bmatrix}, \tag{41}$$

in which $t_d = n_{\mathcal{B}}^2 + 3n_{\mathcal{N}}^2 + n_{\mathcal{B}} n_{\mathcal{N}} + 2n^2 + 2m$. By the interior point condition, the set of solutions of $V(f_d) \cap \Omega_d$ is nonempty, where Ω_d is defined as

$$\Omega_d := \Big\{ \vartheta_d \mid \det(U_X[i]) > 0, \ \ \det(U_{S^a}[j]) > 0, \ \ i = 1, \ldots, n_{\mathcal{B}}, \ j = 1, \ldots, n_{\mathcal{N}} \Big\}. \tag{42}$$

Then, analogous to the primal case, from a solution $\vartheta_d \in V(f_d) \cap \Omega_d$ we can get a solution $(U_{S^a}, y^a, W, Q_{\mathcal{N}})$ with symmetric W, which is the analytic center of the dual optimal set. Therefore, Lemma 8 implies the existence of $\vartheta_d \in V(f_d) \cap \Omega_d$ so that $\|\vartheta_d\| \le r_{\mathcal{D}^*}$, where

$$\begin{aligned} &\log_2(r_{\mathcal{D}^*}) = (L+2)(\bar{t}_d \bar{d}_d)^{\bar{s}_d}, \\ &\bar{d}_d := \max\{n_{\mathcal{B}}, n_{\mathcal{N}}, 3\}, \\ &\bar{t}_d := t_d + n_{\mathcal{B}} + n_{\mathcal{N}} = n_{\mathcal{B}}^2 + 3n_{\mathcal{N}}^2 + n_{\mathcal{B}} n_{\mathcal{N}} + n_{\mathcal{B}} + n_{\mathcal{N}} + 2n^2 + 2m, \\ &\bar{s}_d = n_{\mathcal{B}}^2 + 2n_{\mathcal{N}}^2 + n^2 + n(n_{\mathcal{B}} + n_{\mathcal{N}}) + m. \end{aligned}$$

in which $\bar{s}_d$ and $\bar{d}_d$ are defined analogously as in the primal side. As a result, a lower bound for $\sigma_{\mathcal{N}}$ is given by using $\|W\| \le r_{\mathcal{D}^*}$ and (40). This completes the proof. □

Remark 5. *For the special case $n_{\mathcal{B}} = 0$ we get $\sigma = \sigma_{\mathcal{N}}$ by* (6)*, and thus the lower bound* (29) *is still valid. Indeed, any dual feasible solution is also dual optimal for this special case. Thus, to derive a lower bound for $\sigma_{\mathcal{N}}$ we only need to compute the analytic center of the dual feasible set $\mathcal{D}$, i.e.,*

$$\begin{aligned} \max \quad & \log\big(\det(S^a)\big) \\ \text{s.t.} \quad & \sum_{i=1}^m y_i^a A^i + S^a = C, \\ & S^a \succ 0. \end{aligned} \tag{43}$$

It it easy to verify that the application of Lemma 8 to the system of optimality conditions of (43) *gives an integral polynomial map with strictly fewer number of polynomials and variables than* (41)*, which yields a smaller $r_{\mathcal{D}^*}$.*

Example 2. *From* (29) *we get a doubly exponentially small lower bound for σ, which could be quite loose for some instances. Consider the SDO problem in Example 1 for which we have $\sigma = \frac{2}{3}$. Note that the analytic center of the primal optimal set can be obtained by solving*

$$\max_{0 \le \theta \le 1} \log(\det(Q_{\mathcal{B}}^T X_\theta^* Q_{\mathcal{B}})) = \max_{0 \le \theta \le 1} \log(8\theta - 8\theta^2),$$

which gives the lower bound $\sigma_{\mathcal{B}} \geq 0.4384$. On the other hand, we have $\sigma_{\mathcal{N}} \geq 2$ as the dual optimal set is a singleton. Hence, the analytic center problem yields the lower bound $\sigma \geq 0.4384$.
Given $n_{\mathcal{B}} = 2$, $n_{\mathcal{N}} = 1$, $n = m = 3$, $\|A^1\| = 1$, $\|A^2\| = \|A^3\| = \sqrt{3}$, and $L = l(2) = 1 + \lceil \log_2(3) \rceil = 3$, we can compute the lower bound (29). To that end, we have

$$\bar{t}_p = 30, \quad \bar{t}_d = 36, \quad \bar{s}_p = 24, \quad \bar{s}_d = 27, \quad \bar{d}_p = \bar{d}_d = 3.$$

Therefore, we get $r_{\mathcal{P}^*} = 2^{5\times 90^{24}}$ and $r_{\mathcal{D}^*} = 2^{5\times 108^{27}}$. Consequently,

$$\sigma \geq \min\left\{ \Big(\frac{2\sqrt{3}-1}{11}\Big) 2^{-5\times 90^{24}},\ 2^{-5\times 108^{27}} \right\} = 2^{-5\times 108^{27}},$$

which is quite far from the true value $\sigma = \frac{2}{3}$.

B Error bound for an LMI system

An LMI system is defined as

$$\begin{cases} X \in D_0 + \mathcal{L}, \\ X \succeq 0, \end{cases} \tag{44}$$

where D_0 is a symmetric matrix and $\bar{\mathcal{L}} \subset \mathbb{S}^n$ denotes a linear subspace of symmetric matrices. For system (44), we consider a sequence of solutions denoted by $X(\epsilon)$ for $\epsilon > 0$ which satisfies

$$\mathrm{dist}(X(\epsilon), D_0 + \mathcal{L}) \leq \epsilon, \quad \lambda_{\min}(X(\epsilon)) \geq -\epsilon, \tag{45}$$

for all $\epsilon > 0$, where dist(.) denotes the distance function with respect to a norm. Further, $\bar{\mathcal{L}}$ is defined as the smallest subspace containing $D_0 + \mathcal{L}$, i.e.,

$$\bar{\mathcal{L}} := \{X \in \mathbb{S}^n \mid X + \beta D_0 \in \mathcal{L} \ \text{ for some } \beta\}.$$

The following lemma is in order.

Lemma 10 (Theorem 3.3 in [26]). *Let $\{X(\epsilon) \mid 0 < \epsilon \leq 1\}$ be a set of solutions so that $\|X(\epsilon)\|$ is bounded and (45) holds for all $0 < \epsilon \leq 1$. Then we have*

$$\mathrm{dist}\Big(X(\epsilon), (D_0 + \mathcal{L}) \cap \mathbb{S}^n_+\Big) \leq c\epsilon^{\gamma},$$

where c is a positive condition number, and $\gamma = 2^{-d_s}$ in which d_s denotes the degree of singularity of the linear subspace $\bar{\mathcal{L}}$.

Lemma 11 (Theorem 3.6 in [26]). *For a linear subspace $\bar{\mathcal{L}} \subset \mathbb{S}^n$, we have*

$$d_s \leq \min\big\{n-1, \dim(\bar{\mathcal{L}}), \dim(\bar{\mathcal{L}}^{\perp})\big\}.$$

Example 3. *We can show that the upper bound given in Lemma 11 is indeed tight. To do so, consider the following LMI system*

$$\begin{cases} X_{11} = 0, \\ X_{kk} = x_{1,k+1}, \quad k = 2, \ldots, n-1, \\ X \succeq 0, \end{cases}$$

where the set of feasible solutions is given by

$$X = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & X_{nn} \end{bmatrix}, \qquad X_{nn} \geq 0.$$

Using the facial reduction procedure in [26], we can see that the number of facial reduction steps is $n-1$ *for all* $n \geq 2$*. Due to the lengthy discussion, we omit the details here and refer the interested reader to [21] for a simple demonstration of the facial reduction algorithm. Additional examples of the facial reduction for SDO problems can be found in [3].*