On Identification of the Optimal Partition of Second Order Cone Optimization Problems

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ON IDENTIFICATION OF THE OPTIMAL PARTITION OF SECOND ORDER CONE OPTIMIZATION PROBLEMS

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Abstract. This paper discusses the identification of the optimal partition of second order cone optimization (SOCO). By giving definitions of two condition numbers which only dependent on the SOCO problem itself, we derive some bounds on the magnitude of the blocks of variables along the central path, and prove that the optimal partition $\mathcal{B}$, $\mathcal{N}$, $\mathcal{R}$, and $\mathcal{T}$ for SOCO problems can be identified along the central path when the barrier parameter μ is small enough. Then we generalize the results to a specific neighborhood of the central path.

Key words: Second Order Cone Optimization; Optimal Partition; Convergence Properties of Central Path

MSC: 90C30

1. Introduction . The notion of optimal partition is well known for linear optimization (LO) and linear complementarity problems (LCP). It is an important tool both in identifying exact optimal solutions and in sensitivity analysis, see e.g., [10, 20]. Using a geometric approach, Yildirim [26] extends the concept of optimal partition to general convex conic optimization, and [3] provides another algebraic definition of the optimal partition $\mathcal{B},\mathcal{N},\mathcal{R},\mathcal{T}$ for Second Order Cone Optimization (SOCO). However, as pointed out in [23], the identification of the optimal partition along the central path is still a missing element of the interior point methods (IPM) theory for SOCO.

The identification of optimal partition in IPMs methods is closely related to the limiting behavior of the central path. The analyticity of the central path at the limit has been studied extensively for LO, see, e.g., [1, 5, 7, 24]. The limiting behavior of the central path for LCP as the barrier parameter $\mu \to 0_+$ (where $\mu \to 0_+$ means that $\mu \to 0, \mu > 0$) have been studied e.g., in [8, 19, 21, 22, 16]. For $P_*(\kappa)$ LCPs, the paper [8] proposed a strongly polynomial rounding procedure yielding a maximally complementary solution. The properties of the central path for semidefinite optimization (SDO) problems have been studied by e.g., by [4, 6, 12, 13, 15, 17, 18], where the analyticity of the central path at zero are obtained when the strict complementarity condition is satisfied. However, as pointed out in [23], the convergence properties of the central path of SOCO, and the identification of the optimal partition are not sufficiently studied yet for the general case.

This paper is organized as follows. In Section 2, we review some key results for SOCO. In Section 3, after reviewing the definition of optimal partition for SOCO, we first propose two condition numbers σ_1, σ_2, which are positive constants that dependent only on the optimization problem. Then we derive quantitative results on the magnitude of the variables along the central path, and prove that the optimal parti-

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tion $\mathcal{B}, \mathcal{N}, \mathcal{R}, \mathcal{T}$, proposed by [3], can be identified exactly. In Section 4 we generalize the results derived in Section 3 to the vicinity of the central path and show that if (x, y, s) is given in an appropriate neighborhood of the central path $(x(\mu), y(\mu), s(\mu))$, with μ small enough, we also have a complete separation of the blocks of variables according to the optimal partition. We conclude this paper with some remarks in Section 5.

Notation: In this paper $\|\cdot\|$ denotes the Euclidean 2-norm in R^n, i.e., $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ for $x \in R^n$; $x^T s$ denotes the standard inner product for $x, s \in R^n$, i.e., $x^T s = \sum_{i=1}^n x_i\, s_i$. As in MATLAB, we use "," for stacking vectors and matrices in a row, and use ";" for stacking them in a column. Subscript expressions involving colons refer to portions of a vector or a matrix. For example, $(a; b) = (a^{\mathrm{T}}, b^{\mathrm{T}})^{\mathrm{T}}$, and $x_{2:k} = (x_2, \ldots, x_k)^{\mathrm{T}}$, where "T" indicates the transpose of a vector or a matrix.

2. Preliminaries . SOCO has been studied extensively [14, 2] in the past two decades. Theoretically, SOCO can be seen as a special case of SDO, see, e.g., [25, 2]. However, as pointed out in e.g., in [2], due to its broad applicability, its special structure, high efficiency of IPMs in computational practice, and its theoretical complexity bound, SOCO is worth studying on its own right.

The convex cone

$$\mathcal{K} = \{x = (x_1, \ldots, x_n) \in R^n \mid x_1 \geq \|x_{2:n}\|\}$$

is referred to as a second-order cone (SOC), or Lorentz cone, or quadratic cone. It is well known that the SOC is self-dual, i.e., we have $\mathcal{K} = \mathcal{K}^*$, where

$$\mathcal{K}^* = \{s \in R^n \mid s^T x \geq 0, \ \forall x \in \mathcal{K}\}$$

is the dual cone of $\mathcal{K}$. Denote $\mathcal{K}_q^i = \{x^i = (x_1^i, \ldots, x_{n_i}^i)^{\mathrm{T}} \in R^{n_i} \mid x_1^i \geq \|x_{2:n_i}^i\|\}$ for $i = 1, \ldots, k$. Then the standard form SOCO problem is as follows:

$$\begin{array}{ll} \min & \sum\limits_{i=1}^{k} (c^i)^{\mathrm{T}} x^i \\ \text{s.t.} & \sum\limits_{i=1}^{k} A^i x^i = b, \\ & x^i \in \mathcal{K}_q^i, \ i = 1, 2, \ldots, k, \end{array} \tag{2.1}$$

where $b = (b_1, \ldots, b_m)^{\mathrm{T}} \in R^m$, $A^i \in R^{m \times n_i}$ and $c^i = (c_1^i, c_2^i, \ldots, c_{n_i}^i)^{\mathrm{T}} \in R^{n_i}$ for $i = 1, \ldots, k$. Since for every $i = 1, 2, \ldots, k$, the set $\mathcal{K}_q^i$ is self-dual, i.e., we have $(\mathcal{K}_q^i)^* = \mathcal{K}_q^i$, the corresponding dual of problem (2.1) is:

$$\begin{array}{ll} \max & b^{\mathrm{T}} y \\ \text{s.t.} & (A^i)^{\mathrm{T}} y + s^i = c^i, \ i = 1, \ldots, k, \\ & s^i \in (\mathcal{K}_q^i)^* = \{s^i \mid s_1^i \geq \|s_{2:n_i}^i\|\}, \ i = 1, 2, \ldots, k, \end{array} \tag{2.2}$$

where $y = (y_1, \ldots, y_m)^{\mathrm{T}} \in R^m$ is the dual variable, and $s^i = (s_1^i, \ldots, s_{n_i}^i)^{\mathrm{T}} \in R^{n_i}$ are the slack variables for $i = 1, 2, \ldots, k$.

For brevity let $n = n_1 + n_2 + \cdots + n_k$, and denote $A = [A^1, A^2, \ldots, A^k] \in R^{m \times n}$, $K = \mathcal{K}_q^1 \times \mathcal{K}_q^2 \times \ldots \times \mathcal{K}_q^k$, $c = (c^1; c^2; \ldots; c^k) = (c_1^1, \ldots, c_{n_1}^1, c_1^2, \ldots, c_{n_2}^2, \ldots, c_1^k, \ldots, c_{n_k}^k)^{\mathrm{T}}$, and $x = (x^1; x^2; \ldots; x^k) = (x_1^1, \ldots, x_{n_1}^1, x_1^2, \ldots, x_{n_2}^2, \ldots, x_1^k, \ldots, x_{n_k}^k)^{\mathrm{T}}$. By definition $\mathcal{K}$ is the Cartesian product of several SOCs, hence $\mathcal{K}$ is also self-dual, i.e., we have

$\mathcal{K}^* = \mathcal{K}$. By $x \succeq_{\mathcal{K}} 0$ ($x \succ_{\mathcal{K}} 0$), where $x \in R^n$, we mean that $x \in \mathcal{K}$ ($x \in \text{int}(\mathcal{K})$). Then the SOCO problem (2.1) and its dual (2.2), analogous to LO, can also be written as

$$(P) \quad \begin{array}{ll} \min & c^{\mathrm{T}}x \\ \text{s.t.} & Ax = b, \\ & x \succeq_{\mathcal{K}} 0. \end{array} \qquad (D) \quad \begin{array}{ll} \max & b^{\mathrm{T}}y \\ \text{s.t.} & A^T y + s = c, \\ & s \succeq_{\mathcal{K}} 0. \end{array} \tag{2.3}$$

In order to analyze the properties of problem (2.3), the following two standard assumptions are made.

Assumption 1. Matrix $A = [A^1, A^2, \ldots, A^k] \in R^{m\times n}$ in (2.3) has full row rank, i.e., $\text{rank}(A) = m$.

Assumption 2. Both the primal problem (P) and the dual problem (D) in (2.3) have strictly feasible solutions, i.e.,

$$\exists x \in \text{int}(\mathcal{K}) \text{ such that } Ax = b$$

$$\exists (y, s) \in R^m \times \text{int}(\mathcal{K}) \text{ such that } A^T y + s = c.$$

Assumption 1 is a technical one. It enforces a one-to-one correspondence between y and s for dual solutions (y, s). Therefore, when the solution s is bounded, so is the corresponding solution y. On the other hand, Assumption 2 is a Slater condition, which is essential in the development of the theory of convex optimization.

Now let us introduce the customary notation in SOCO:

$$x \circ s = \begin{pmatrix} x^{\mathrm{T}}s \\ x_1 s_{2:n} + s_1 x_{2:n} \end{pmatrix},$$

where $x = (x_1; x_{2:n}) = (x_1, x_2 \ldots, x_n)^{\mathrm{T}}$ and $s = (s_1; s_{2:n}) = (s_1, s_2 \ldots, s_n)^{\mathrm{T}}$. For $x = (x^1; \ldots; x^k) \in \mathcal{K}$, $s = (s^1; \ldots; s^k) \in \mathcal{K}$, where $x^i, s^i \in \mathcal{K}_q^i$ for $i = 1, \ldots, k$, define

$$x \circ s = (x^1 \circ s^1; \ldots; x^k \circ s^k).$$

Denote $\mathcal{F}$ as the set of all primal-dual feasible points for (2.3), $\mathcal{F}^*$ as the set of all primal-dual optimal solutions for (2.3), i.e., we have

$$\begin{aligned} \mathcal{F} = \{(x, y, s) \in R^n \times R^m \times R^n \mid\; & x \text{ is feasible for the primal problem } (P) \text{ in (2.3)}, \\ & (y, s) \text{ is feasible for the dual problem } (D) \text{ in (2.3)}\} \\ \mathcal{F}^* = \{(x, y, s) \in R^n \times R^m \times R^n \mid\; & x \text{ is optimal for the primal problem } (P) \text{ in (2.3)}, \\ & (y, s) \text{ is optimal for the dual problem } (D) \text{ in (2.3)}\} \end{aligned}$$

Suppose that $\mathcal{K}_q$ is a second order cone. It is well known that for all $x, s \in \mathcal{K}_q$, we have $x^{\mathrm{T}}s \geq 0$, and that $x^{\mathrm{T}}s = 0$ is equivalent to $x \circ s = 0$. We have the following results for the primal-dual pair of SOCO problems (2.3) (see, e.g., [14, 2]).

THEOREM 2.1. *Consider the SOCO problem (P) and its dual (D) as in (2.3).*

1. *If $(x, y, s) \in \mathcal{F}$, then the duality gap $c^{\mathrm{T}}x - b^{\mathrm{T}}y = s^{\mathrm{T}}x \geq 0$.*
2. *If Assumption 2 is satisfied, then both the primal and the dual problems in (2.3) have optimal solutions x^*, (y^*, s^*) and $c^{\mathrm{T}}x^* = b^{\mathrm{T}}y^*$, i.e., the duality gap $(x^*)^{\mathrm{T}}s^* = 0$, which is equivalent to $x^* \circ s^* = 0$ for $x^* \in \mathcal{K}$ and $s^* \in \mathcal{K}$. Moreover, a point $(x, y, s) \in \mathcal{F}^*$, if and only if*

$$\begin{aligned} Ax = b, &\quad x \in \mathcal{K}, \\ A^{\mathrm{T}}y + s = c, &\quad s \in \mathcal{K},\; y \in R^m, \\ x \circ s = 0, & \end{aligned} \tag{2.4}$$

where $x = (x^1; \ldots; x^k) \in \mathcal{K}$ and $s = (s^1; \ldots; s^k) \in \mathcal{K}$ with $x^i, s^i \in \mathcal{K}_q^i$.

3. *If both Assumption 1 and Assumption 2 are satisfied, then the optimal solution set $\mathcal{F}^*$ of (2.3) is a nonempty and compact convex set.*

Now we give the definition of the central path. As in [2], we denote $e_k = (1;0;...;0) \in R^k$, $e = (e_{n_1};e_{n_2};\ldots;e_{n_k})$, where " ; " is a concatenation operation for vectors and matrices in columns. The central path for problem (2.3) is defined as the set of solutions $(x(\mu), y(\mu), s(\mu))$, where $\mu > 0$, of the following system:

$$\begin{aligned} Ax &= b, \quad x \in \mathcal{K}, \\ A^{\mathrm{T}}y + s &= c, \quad s \in \mathcal{K},\ y \in R^m, \\ x \circ s &= \mu e. \end{aligned} \tag{2.5}$$

System (2.5) can be seen as a perturbation of system (2.4). We have the following result (see, e.g., [14, 2, 9]):

THEOREM 2.2. *If both Assumption 1 and Assumption 2 are satisfied, we have:*

1. *For any $\mu > 0$ system (2.5) has a unique solution $(x(\mu), y(\mu), s(\mu))$. Moreover, we have $x^i(\mu) \in \operatorname{int}(\mathcal{K}_q^i)$ and $s^i(\mu) \in \operatorname{int}(\mathcal{K}_q^i)$, for every $i = 1, \ldots, k$.*
2. *For $\mu > 0$, the sequence $(x(\mu), y(\mu), s(\mu))$ defines a vector-valued analytical function of μ.*
3. *The sequence $(x(\mu), y(\mu), s(\mu))$ converges to a maximally complementary optimal solution $(x^*, y^*, z^*) \in \mathcal{F}^* \subset R^n \times R^m \times R^n$ of (2.3) as $\mu \to 0_+$, where $\mu \to 0_+$ means that $\mu \to 0$ while $\mu > 0$.*

Theorem 2.2 tells us that the central path $\{(x(\mu), y(\mu), s(\mu)) \mid \mu > 0\}$ is properly defined, and for $\mu > 0$ it is a smooth analytical curve in $R^n \times R^m \times R^n$. In this paper we study the properties of the central path when $\mu \to 0_+$. Unlike the case of Linear Optimization (LO), the central path is not differentiable at zero for general SOCO problems.

3. The identification of the optimal partition. The optimal partition for the primal-dual SOCO problem pair (2.3) consists of four sets, which are defined in [3] as (see also [23]):

$$\begin{aligned}
\mathcal{B} &= \{i \mid x_1^i > \|x_{2:n_i}^i\| \text{ for a primal optimal solution } x\}, \\
\mathcal{N} &= \{i \mid s_1^i > \|s_{2:n_i}^i\| \text{ for a dual optimal solution } (y,s)\}, \\
\mathcal{R} &= \{i \mid x_1^i = \|x_{2:n_i}^i\| > 0,\ s_1^i = \|s_{2:n_i}^i\| > 0 \\
&\qquad \text{for a primal-dual optimal solution } (x,y,s)\}, \\
\mathcal{T} &= \{i \mid x^i = s^i = 0;\ \text{or } x^i = 0,\ s_1^i = \|s_{2:n_i}^i\| > 0;\ \text{or} \\
&\qquad x_1^i = \|x_{2:n_i}^i\| > 0,\ s^i = 0 \text{ for all primal-dual optimal solutions } (x,y,s)\}.
\end{aligned}$$

It is obvious, due to the convexity of the optimal set, that the sets $\mathcal{B}, \mathcal{N}, \mathcal{R}$, and $\mathcal{T}$ are disjoint and $\mathcal{B} \cup \mathcal{N} \cup \mathcal{R} \cup \mathcal{T} = \{1, 2, \ldots, k\}$.

In the following analysis, we will always assume that both Assumption 1 and Assumption 2 are satisfied.

LEMMA 3.1. *For $\forall i \in \mathcal{B} \cup \mathcal{N} \cup \mathcal{R}$, we have:*

1. *If $i \in \mathcal{B}$, we have $s^i = 0$ for $\forall\ (x,y,s) \in \mathcal{F}^*$.*
2. *If $i \in \mathcal{N}$, we have $x^i = 0$ for $\forall\ (x,y,s) \in \mathcal{F}^*$.*
3. *If $i \in \mathcal{R}$, then for every $(x,y,s) \in \mathcal{F}^*$, we can write:*

$$x^i = \alpha \begin{pmatrix} 1 \\ h \end{pmatrix}, \quad s^i = \beta \begin{pmatrix} 1 \\ -h \end{pmatrix},$$

where $h = \frac{x^i_{2:n_i}}{\|x^i_{2:n_i}\|} = -\frac{s^i_{2:n_i}}{\|s^i_{2:n_i}\|} \in R^{n_i-1}$ *is a constant vector for all optimal solutions* $(x, y, s) \in \mathcal{F}^*$ *with* $\|h\| = 1$*, while* $\alpha = x^i_1 \geq 0, \beta = s^i_1 \geq 0$ *may change with the particular optimal solution* $(x, y, s) \in \mathcal{F}^*$*.*

Proof. By the optimality conditions (2.4) in Theorem 2.1, we know that for $\forall\ (x, y, s) \in \mathcal{F}^*$ and $\forall\ (\bar{x}, \bar{y}, \bar{s}) \in \mathcal{F}^*$, we have $(x^i)^{\mathrm{T}}\bar{s}^i = 0$ and $(\bar{x}^i)^{\mathrm{T}}s^i = 0$ for all $i = 1, 2, \dots, k$, where $x = (x^1; \dots; x^k), s = (s^1; \dots; s^k)$ with $x^i, \bar{x}^i, s^i, \bar{s}^i \in \mathcal{K}^i_q \subset R^{n_i}$ for $i = 1, \dots, k$.

1. Since $i \in \mathcal{B}$, there exists some $(\bar{x}, \bar{y}, \bar{s}) \in \mathcal{F}^*$ such that $\bar{x}^i \in \mathrm{int}(\mathcal{K}^i_q)$, i.e., $\bar{x}^i_1 > \|\bar{x}^i_{2:n_i}\| \geq 0$. Since for all $(x, y, s) \in \mathcal{F}^*$, we have $s^i_1 \geq \|s^i_{2:n_i}\|$ and $(\bar{x}^i)^{\mathrm{T}}s^i = 0$. By the Cauchy-Schwarz inequality we get

$$0 = (\bar{x}^i)^{\mathrm{T}}s^i = \bar{x}^i_1 s^i_1 + \sum_{j=2}^{n_i} \bar{x}^i_j s^i_j \geq \bar{x}^i_1 s^i_1 - \|\bar{x}^i_{2:n_i}\|\,\|s^i_{2:n_i}\| \geq 0. \tag{3.1}$$

   Therefore we have $\bar{x}^i_1 s^i_1 - \|\bar{x}^i_{2:n_i}\|\,\|s^i_{2:n_i}\| = 0$. Then, by $s^i_1 \geq \|s^i_{2:n_i}\|$ and $\bar{x}^i_1 > \|\bar{x}^i_{2:n_i}\|$, we get $s^i_1 = \|s^i_{2:n_i}\| = 0$, which is equivalent to $s^i = 0$.
2. In the same way as above we can get the desired result.
3. Since $i \in \mathcal{R}$, by the definition of $\mathcal{R}$, we know that there exists some $(\bar{x}, \bar{y}, \bar{s}) \in \mathcal{F}^*$ such that $\bar{x}^i_1 = \|\bar{x}^i_{2:n_i}\| > 0$ and $\bar{s}^i_1 = \|\bar{s}^i_{2:n_i}\| > 0$. For $\forall\ (x, y, s) \in \mathcal{F}^*$, we have $x^i_1 \geq \|x^i_{2:n_i}\|, s^i_1 \geq \|s^i_{2:n_i}\|$ and $(\bar{x}^i)^{\mathrm{T}}s^i = 0$. Then, by (3.1), we get $s^i_1 = \|s^i_{2:n_i}\|$ and

$$(\bar{x}^i_{2:n_i})^{\mathrm{T}}s^i_{2:n_i} = -\|\bar{x}^i_{2:n_i}\|\,\|s^i_{2:n_i}\|.$$

   By the equality conditions of the Cauchy-Schwarz inequality and $\|\bar{x}^i_{2:n_i}\| > 0$, there exists some $\tilde{\beta} \geq 0$ such that $s^i_{2:n_i} = -\tilde{\beta}\bar{x}^i_{2:n_i}$. Since $\|\bar{x}^i_{2:n_i}\| = \bar{x}^i_1 > 0, \|s^i_{2:n_i}\| = s^i_1 \geq 0$, we get $\tilde{\beta} = \frac{s^i_1}{\bar{x}^i_1} \geq 0$, and hence we have

$$s^i_{2:n_i} = -\tilde{\beta}\bar{x}^i_{2:n_i} = -\frac{s^i_1}{\bar{x}^i_1}\bar{x}^i_{2:n_i}. \tag{3.2}$$

   Define

$$h = \frac{\bar{x}^i_{2:n_i}}{\|\bar{x}^i_{2:n_i}\|} = \frac{\bar{x}^i_{2:n_i}}{\bar{x}^i_1}.$$

   By (3.2) we get

$$s^i = \begin{pmatrix} s^i_1 \\ s^i_{2:n_i} \end{pmatrix} = \frac{s^i_1}{\bar{x}^i_1}\begin{pmatrix} \bar{x}^i_1 \\ -\bar{x}^i_{2:n_i} \end{pmatrix} = s^i_1\begin{pmatrix} 1 \\ -h \end{pmatrix} = \beta\begin{pmatrix} 1 \\ -h \end{pmatrix},$$

   where $\beta = s^i_1 \geq 0$ and $h \in R^{n_i-1}$ is a constant vector (which is independent of (x, y, s)) with $\|h\| = 1$.
   According to the optimality conditions, we have $(\bar{x}^i)^{\mathrm{T}}\bar{s}^i = 0$, hence in the same way as above we get

$$\bar{s}^i = \frac{\bar{s}^i_1}{\bar{x}^i_1}\begin{pmatrix} \bar{x}^i_1 \\ -\bar{x}^i_{2:n_i} \end{pmatrix} = \bar{s}^i_1\begin{pmatrix} 1 \\ -h \end{pmatrix}.$$

Since $(\bar{s}^i)^{\mathrm{T}} x^i = 0$, the same way we get

$$x^i = \frac{x_1^i}{\bar{s}_1^i} \begin{pmatrix} \bar{s}_1^i \\ -\bar{s}_{2:n_i}^i \end{pmatrix} = x_1^i \begin{pmatrix} 1 \\ h \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ h \end{pmatrix},$$

where $\alpha = x_1^i \geq 0$.

6 $\square$

Denote $\mathrm{bd}(\mathcal{K}_q^i) = \{x^i \in \mathcal{K}_q^i \mid x_1^i = \|x_{2:n_i}\| > 0\}$, and $\mathrm{int}(\mathcal{K}_q^i) = \{x^i \in \mathcal{K}_q^i \mid x_1^i > \|x_{2:n_i}\|\}$. Then each block x^i may be in one of the following three states: $x^i \in \mathrm{int}(\mathcal{K}_q^i)$, or $x^i \in \mathrm{bd}(\mathcal{K}_q^i)$, or $x^i = 0$. According to Lemma 3.1, it is impossible to have both $x^i \in \mathrm{int}(\mathcal{K}_q^i)$ and $s^i \in \mathrm{int}(\mathcal{K}_q^i) \cup \mathrm{bd}(\mathcal{K}_q^i)$, or both $s^i \in \mathrm{int}(\mathcal{K}_q^i)$ and $x^i \in \mathrm{int}(\mathcal{K}_q^i) \cup \mathrm{bd}(\mathcal{K}_q^i)$. However, if we have $\bar{x}^i = 0$ and $\bar{s}^i \in \mathrm{bd}(\mathcal{K}_q^i) \cup \{0\}$, or $\bar{s}^i = 0$ and $\bar{x}^i \in \mathrm{bd}(\mathcal{K}_q^i) \cup \{0\}$ for some optimal solution $(\bar{x}, \bar{y}, \bar{s}) \in \mathcal{F}^*$, then there may still exist some other optimal solution $(x, y, s) \in \mathcal{F}^*$ with $x^i \in \mathrm{bd}(\mathcal{K}_q^i)$ and $s^i \in \mathrm{bd}(\mathcal{K}_q^i)$, and vice versa. Hence, in such a case, we have $i \in \mathcal{R}$, and so $i \notin \mathcal{T}$. Now, as pointed out in [23], we can enumerate all the possible configurations for the primal-dual blocks of variables at optimality. These configurations are listed in Table 3.1, where cases that are not possible are indicated by "$\times$".

TABLE 3.1
Possible configurations for the i^{th} blocks in an optimal solution.

s^i \ x^i	0	$\mathrm{bd}(\mathcal{K}_q^i)$	$\mathrm{int}(\mathcal{K}_q^i)$
0	$i \in \mathcal{T} \cup \mathcal{R}$	$i \in \mathcal{T} \cup \mathcal{R}$	$i \in \mathcal{B}$
$\mathrm{bd}(\mathcal{K}_q^i)$	$i \in \mathcal{T} \cup \mathcal{R}$	$i \in \mathcal{R}$	$\times$
$\mathrm{int}(\mathcal{K}_q^i)$	$i \in \mathcal{N}$	$\times$	$\times$

One can see that the set $\mathcal{T}$ is complementary to $\mathcal{B} \cup \mathcal{N} \cup \mathcal{R}$ by definition, and the intersection of any pair of the three sets $\mathcal{B}, \mathcal{N}, \mathcal{R}$ is empty by Lemma 3.1. Hence, as in [3], we have the following result.

COROLLARY 3.2. *The four sets $\mathcal{B}, \mathcal{N}, \mathcal{R}, \mathcal{T}$, defined by the optimal solution set $\mathcal{F}^*$, give a partition of the index set $\{1, \ldots, k\}$*

In order to derive bounds for the magnitude of the variables $(x(\mu), y(\mu), s(\mu))$ along the central path as $\mu \to 0_+$, for SOCO problems we define two condition numbers σ_1 and σ_2 as follows:

$$\sigma_B = \min_{i \in \mathcal{B}} \max_{(x,y,s) \in \mathcal{F}^*} \{x_1^i - \|x_{2:n_i}^i\|\}, \tag{3.3}$$

$$\sigma_N = \min_{i \in \mathcal{N}} \max_{(x,y,s) \in \mathcal{F}^*} \{s_1^i - \|s_{s:n_i}^i\|\}, \tag{3.4}$$

$$\sigma_1 = \min\{\sigma_B, \sigma_N\}, \tag{3.5}$$

$$\sigma_2 = \min_{i \in \mathcal{R}} \max_{(x,y,s) \in \mathcal{F}^*} \{x_1^i + s_1^i - \|x_{s:n_i}^i + s_{2:n_i}^i\|\}. \tag{3.6}$$

By Lemma 3.1 and definitions (3.3)–(3.5), we define

$$\sigma_1 = \min_{i \in \mathcal{B} \cup \mathcal{N}} \max_{(x,y,s) \in \mathcal{F}^*} \{x_1^i + s_1^i - \|x_{2:n_i}^i + s_{2:n_i}^i\|\}. \tag{3.7}$$

Observe, that the definitions of the two condition numbers σ_1 and σ_2 have the same form, only that the index sets are different. When Assumptions 1 and 2 are satisfied, then the set of optimal solutions $\mathcal{F}^*$ is nonempty, convex and compact. Thus, the two condition numbers σ_1 and σ_2 are well defined, which is spelled out in the following Lemma.

LEMMA 3.3. *The two condition numbers σ_1 and σ_2 are both positive constants, i.e., we have $\sigma_1 > 0$, and $\sigma_2 > 0$.*

Proof. By the compactness of $\mathcal{F}^*$ and the definitions of σ_1 and σ_2, it is obvious that they both are constants. Further, for $\forall i \in B$, there exists some $(\bar{x}, \bar{y}, \bar{s}) \in \mathcal{F}^*$ such that $\bar{x}_1^i - \|\bar{x}_{2:n_i}^i\| > 0$. Since by Theorem 2.1 $\mathcal{F}^*$ is nonempty and compact, and $x_1^i - \|x_{2:n_i}^i\|$ is a continuous function on the compact set $\mathcal{F}^*$, there must exist some $(\hat{x}, \hat{y}, \hat{s}) \in \mathcal{F}^*$ such that

$$\sigma_1^i := \max_{(x,y,s)\in\mathcal{F}^*} \{x_1^i - \|x_{2:n_i}^i\|\} = \hat{x}_1^i - \|\hat{x}_{2:n_i}^i\| \geq \bar{x}_1^i - \|\bar{x}_{2:n_i}^i\| > 0.$$

Then by the finiteness of the set $\mathcal{B}$ we obtain $\sigma_B = \min_{i\in\mathcal{B}} \sigma_1^i > 0$. In the same way we can prove $\sigma_N > 0$, and hence $\sigma_1 = \min\{\sigma_B, \sigma_N\} > 0$.

Similarly, for $\forall i \in R$, there exists some $(\bar{x}, \bar{y}, \bar{s}) \in \mathcal{F}^*$ such that $\bar{x}_1^i = \|\bar{x}_{2:n_i}^i\| > 0$ and $\bar{s}_1^i = \|\bar{s}_{2:n_i}^i\| > 0$. Then by Lemma 3.1 we have

$$\bar{x}^i = \bar{x}_1^i \begin{pmatrix} 1 \\ h_i \end{pmatrix}, \quad \bar{s}^i = \bar{s}_1^i \begin{pmatrix} 1 \\ -h_i \end{pmatrix},$$

where $h_i \in R^{n_i - 1}$ is a constant vector with $\|h_i\| = 1$. So we have

$$\bar{x}_1^i + \bar{s}_1^i - \|\bar{x}_{2:n_i}^i + \bar{s}_{2:n_i}^i\| = \bar{x}_1^i + \bar{s}_1^i - |\bar{x}_1^i - \bar{s}_1^i| = 2\min\{\bar{x}_1^i, \bar{s}_1^i\} > 0.$$

In a similar way, using the compactness of $\mathcal{F}^*$, the continuity of the function $x_1^i + s_1^i - \|x_{2:n_i}^i + s_{2:n_i}^i\|$ on $\mathcal{F}^*$, and the finiteness of the set $\mathcal{R}$, we get that $\sigma_2 > 0$. $\square$

Lemma 3.3 tells us that the two condition numbers σ_1 and σ_2 are well defined finite positive values. By using σ_1 and σ_2, according to the optimal partition $\mathcal{B}$, $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{T}$, we can derive some bounds for the variables along the central path of the SOCO problem.

THEOREM 3.4. *Let $\mu > 0$ and $(x(\mu), y(\mu), s(\mu))$ be the corresponding point on the central path which satisfies (2.5). Then we have*

1. *For $\forall i \in \mathcal{B}$, we have*

$$x_1^i(\mu) \geq x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| > \frac{\sigma_1}{2k}, \quad \text{and} \quad s_1^i(\mu) \leq \frac{k\mu}{\sigma_1}.$$

2. *For $\forall i \in \mathcal{N}$, we have*

$$s_1^i(\mu) \geq s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| > \frac{\sigma_1}{2k}, \quad \text{and} \quad x_1^i(\mu) \leq \frac{k\mu}{\sigma_1}.$$

3. *For $\forall i \in \mathcal{R}$, we have*

$$x_1^i(\mu) > \frac{\sigma_2}{4k}, \quad \text{and} \quad s_1^i(\mu) > \frac{\sigma_2}{4k},$$

$$(x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\|) + (s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\|) \leq \frac{2k\mu}{\sigma_2}.$$

In particular we have

$$\frac{2k\mu}{\sigma_2} > x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\|, \quad \text{and} \quad \frac{2k\mu}{\sigma_2} > s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\|.$$

4. For $\forall i \in \mathcal{B} \cup \mathcal{N}$ we have

$$x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\| > \frac{\sigma_1}{2k}.$$

For $\forall i \in \mathcal{R}$ we have

$$x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\| > \frac{\sigma_2}{2k}.$$

For $\forall i \in \mathcal{T}$, we have

$$x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\| \to 0 \quad as\ \mu \to 0_+.$$

Proof. By (2.5) for any $i \in \{1, \dots, k\}$ we have

$$x^i(\mu) \circ s^i(\mu) = \mu e_i = (\mu, 0, \dots, 0)^{\mathrm{T}}, \tag{3.8}$$

which is equivalent to:

$$s^i(\mu) = \frac{\mu \begin{pmatrix} x_1^i(\mu) \\ -x_{2:n_i}^i(\mu) \end{pmatrix}}{(x_1^i(\mu))^2 - \|x_{2:n_i}^i(\mu)\|^2}, \tag{3.9}$$

or equivalently

$$x^i(\mu) = \frac{\mu \begin{pmatrix} s_1^i(\mu)) \\ -s_{2:n_i}^i(\mu) \end{pmatrix}}{(s_1^i(\mu))^2 - \|s_{2:n_i}^i(\mu)\|^2}. \tag{3.10}$$

1. For $\forall i \in B$, by the definition of σ_1 and the compactness of $\mathcal{F}^*$, we can choose some $(\bar{x}, \bar{y}, \bar{s}) \in \mathcal{F}^*$ such that

$$\bar{x}_1^i - \|\bar{x}_{2:n_i}^i\| \geq \sigma_1. \tag{3.11}$$

Since both $(\bar{x}, \bar{y}, \bar{s})$ and $(x(\mu), y(\mu), s(\mu))$ are primal-dual feasible, we get

$$\begin{aligned} (\bar{x} - x(\mu))^{\mathrm{T}}(\bar{s} - s(\mu)) &= (\bar{x} - x(\mu))^{\mathrm{T}}(A^{\mathrm{T}}\bar{y} - A^{\mathrm{T}}y(\mu)) \\ &= (A\bar{x} - Ax(\mu))^{\mathrm{T}}(\bar{y} - y) = (b - b)^{\mathrm{T}}(\bar{y} - y) = 0. \end{aligned}$$

Therefore we have

$$\bar{x}^{\mathrm{T}}\bar{s} + x(\mu)^{\mathrm{T}}s(\mu) = \bar{x}^{\mathrm{T}}s(\mu) + x(\mu)^{\mathrm{T}}\bar{s}. \tag{3.12}$$

Since $(\bar{x}, \bar{y}, \bar{s}) \in \mathcal{F}^*$, by the optimality conditions in Theorem 2.1 we have $\bar{x}^{\mathrm{T}}\bar{s} = 0$. By formula (3.8) we have $(x^j(\mu))^{\mathrm{T}}s^j(\mu) = \mu$ for $j = 1, 2, \dots, k$, and hence $x(\mu)^{\mathrm{T}}s(\mu) = \sum_{j=1}^k (x^j(\mu))^{\mathrm{T}}s^j(\mu) = k\mu$. Then by formula (3.12) we get

$$\sum_{j=1}^k [(\bar{x}^j)^{\mathrm{T}}s^j(\mu) + (\bar{s}^j)^{\mathrm{T}}x^j(\mu)] = k\mu. \tag{3.13}$$

Since $(\bar{x}^j)^{\mathrm{T}}s^j(\mu) \geq 0,\ (\bar{s}^j)^{\mathrm{T}}x^j(\mu) \geq 0$ and $s_1^j(\mu) > \|s_{2:n_i}^j(\mu)\|$ for $j = 1, \dots, k$, by formula (3.13), the Cauchy-Schwarz inequality and formula (3.11) we get:

$$\begin{aligned} k\mu &\geq (\bar{x}^i)^{\mathrm{T}}s^i(\mu) = \bar{x}_1^i s_1^i(\mu) + (\bar{x}_{2:n_i}^i)^{\mathrm{T}}s_{2:n_i}^i(\mu) \\ &\geq \bar{x}_1^i s_1^i(\mu) - \|\bar{x}_{2:n_i}^i\|\, \|s_{2:n_i}^i(\mu)\| \geq \sigma_1 s_1^i(\mu). \end{aligned}$$

Hence we have

$$s_1^i(\mu) \le \frac{k\mu}{\sigma_1}, \quad \forall i \in B. \tag{3.14}$$

By (3.9) and $x_1^i(\mu) > \|x_{2:n_i}^i(\mu)\|$ we have

$$\begin{aligned} s_1^i(\mu) &= \frac{\mu x_1^i(\mu)}{(x_1^i(\mu))^2 - \|x_{2:n_i}^i(\mu)\|^2} \\ &= \frac{\mu}{x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\|} \, \frac{x_1^i(\mu)}{x_1^i(\mu) + \|x_{2:n_i}^i(\mu)\|} \\ &> \frac{\mu}{x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\|} \, \frac{1}{2}. \end{aligned} \tag{3.15}$$

Then by (3.14) and (3.15) we get

$$x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| > \frac{1}{2} \, \frac{\mu}{s_1^i(\mu)} \ge \frac{\sigma_1}{2k},$$

and it is obvious that $x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| \le x_1^i(\mu)$.

2. Analogously, by substituting $x^i(\mu)$ for $s^i(\mu)$ and $s^i(\mu)$ for $x^i(\mu)$ respectively, we can get the desired result in the same way as above.
3. By the the definition of σ_2 and the compactness of $\mathcal{F}^*$, for all $i \in \mathcal{R}$, we can choose some $(\bar{x}, \bar{y}, \bar{s}) \in \mathcal{F}^*$ such that

$$\bar{x}_1^i + \bar{s}_1^i - \|\bar{x}_{2:n_i}^i + \bar{s}_{2:n_i}^i\| \ge \sigma_2. \tag{3.16}$$

By Lemma 3.1 we have

$$\bar{x}^i = \bar{x}_1^i \begin{pmatrix} 1 \\ h_i \end{pmatrix}, \quad \bar{s}^i = \bar{s}_1^i \begin{pmatrix} 1 \\ -h_i \end{pmatrix}, \tag{3.17}$$

where $h_i \in R^{n_i - 1}$ is a constant vector with $\|h_i\| = 1$. So we have

$$\bar{x}_1^i + \bar{s}_1^i - \|\bar{x}_{2:n_i}^i + \bar{s}_{2:n_i}^i\| = \bar{x}_1^i + \bar{s}_1^i - |\bar{x}_1^i - \bar{s}_1^i| = 2\min\{\bar{x}_1^i, \bar{s}_1^i\}. \tag{3.18}$$

Then by (3.16) and (3.18) we get

$$\bar{x}_1^i \ge \frac{\sigma_2}{2}, \quad \bar{s}_1^i \ge \frac{\sigma_2}{2}. \tag{3.19}$$

By (3.8) we have

$$x_1^i(\mu) s_{2:n_i}^i(\mu) + s_1^i(\mu) x_{2:n_i}^i(\mu) = 0,$$

which, since $x_1^i(\mu) > 0$, is equivalent to:

$$s_{2:n_i}^i(\mu) = -\frac{s_1^i(\mu)}{x_1^i(\mu)} x_{2:n_i}^i(\mu). \tag{3.20}$$

Then by (3.13), (3.17), (3.19), $\|h_i\| = 1$ and the Cauchy-Schwarz inequality we derive

$$\begin{aligned} k\mu &\ge (\bar{x}^i)^{\mathrm{T}} s^i(\mu) + (\bar{s}^i)^{\mathrm{T}} x^i(\mu) \\ &= \bar{x}_1^i (s_1^i(\mu) + (s_{2:n_i}^i(\mu))^{\mathrm{T}} h_i) + \bar{s}_1^i (x_1^i(\mu) - (x_{2:n_i}^i(\mu))^{\mathrm{T}} h_i) \\ &\ge \bar{x}_1^i (s_1^i(\mu) - \|s_{2:n_i}^i\|) + \bar{s}_1^i (x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\|) \\ &\ge \frac{\sigma_2}{2} (s_1^i(\mu) - \|s_{2:n_i}^i\|) + \frac{\sigma_2}{2} (x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\|). \end{aligned} \tag{3.21}$$

So we get

$$\frac{2k\mu}{\sigma_2} \geq (s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\|) + (x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\|).$$

Since $s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| > 0$ and $x_1^i(\mu) - \|x_{2:n_i}^i(\mu) > 0$, we get

$$\frac{2k\mu}{\sigma_2} > x_1^i(\mu) - \|x_{2:n_i}^i\|, \quad \frac{2k\mu}{\sigma_2} > s_1^i(\mu) - \|s_{2:n_i}^i\|. \tag{3.22}$$

Then by (3.15) and (3.22) we have

$$s_1^i(\mu) \geq \frac{\mu}{2(x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\|)} > \frac{\sigma_2}{4k}.$$

Analogously, by (3.10) and $s_1^i(\mu) > \|s_{2:n_i}^i(\mu)\|$ we have

$$\begin{aligned} x_1^i(\mu) &= \frac{\mu s_1^i(\mu)}{(s_1^i(\mu))^2 - \|s_{2:n_i}^i(\mu)\|^2} \\ &= \frac{\mu}{s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\|} \, \frac{s_1^i(\mu)}{s_1^i(\mu) + \|s_{2:n_i}^i(\mu)\|} \\ &> \frac{\mu}{s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\|} \, \frac{1}{2}. \end{aligned} \tag{3.23}$$

Then by (3.22) and (3.23) we get

$$x_1^i(\mu) \geq \frac{\mu}{2(s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\|)} > \frac{\sigma_2}{4k}.$$

4. Now by the results in item 1 of this theorem, for all $i \in \mathcal{B}$ we have

$$\begin{aligned} & x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\| \\ & \quad \geq x_1^i(\mu) + s_1^i(\mu) - (\|x_{2:n_i}^i(\mu)\| + \|s_{2:n_i}^i(\mu)\|) \\ & \quad > x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| \geq \frac{\sigma_1}{2k}. \end{aligned} \tag{3.24}$$

Similarly for all $i \in \mathcal{N}$ we have

$$\begin{aligned} & x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\| \\ & \quad \geq x_1^i(\mu) + s_1^i(\mu) - (\|x_{2:n_i}^i(\mu)\| + \|s_{2:n_i}^i(\mu)\|) \\ & \quad > s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| \geq \frac{\sigma_1}{2k}. \end{aligned} \tag{3.25}$$

Then by (3.24)–(3.25), for all $i \in \mathcal{B} \cup \mathcal{N}$ we have

$$x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\| > \frac{\sigma_1}{2k},$$

For all $i \in \mathcal{R}$, by (3.20) and the results in item 3 of this theorem, we have

$$\begin{aligned} & x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\| \\ & \quad = x_1^i(\mu) + s_1^i(\mu) - \left|1 - \frac{s_1^i(\mu)}{x_1^i(\mu)}\right| \|x_{2:n_i}^i(\mu)\| \\ & \quad \geq x_1^i(\mu) + s_1^i(\mu) - \left|1 - \frac{s_1^i(\mu)}{x_1^i(\mu)}\right| x_1^i(\mu) \\ & \quad \geq 2\min\{x_1^i(\mu), s_1^i(\mu)\} > \frac{\sigma_2}{2k}. \end{aligned} \tag{3.26}$$

By Theorem 2.2, we know that for all $i \in \mathcal{T}$, we have either $x^i(\mu) \to 0$ and $s^i(\mu) \to 0$, or $x^i(\mu) \to 0$ and $s^i_1(\mu) - s^i_{2:n_i}(\mu) \to 0$, or $s^i(\mu) \to 0$ and $x^i_1(\mu) - x^i_{2:n_i}(\mu) \to 0$ as $\mu \to 0_+$. So we get

$$x^i_1(\mu) + s^i_1(\mu) - \|x^i_{2:n_i}(\mu) + s^i_{2:n_i}(\mu)\| \to 0 \quad \text{as } \mu \to 0_+. \tag{3.27}$$

$\square$

By Theorem 3.4, we get the following known result as a corollary.

COROLLARY 3.5. *The central path $(x(\mu), y(\mu), s(\mu))$ of SOCO problem (2.3) converges to a maximally complementary optimal solution $(\bar{x}, \bar{y}, \bar{s})$.*

Proof. By Theorem 2.2 we have $(x(\mu), y(\mu), s(\mu)) \to (\bar{x}, \bar{y}, \bar{s}) \in \mathcal{F}^*$ as $\mu \to 0_+$. Then by Theorem 3.4 for all $i \in \mathcal{B}$ we have $\bar{x}^i_1 - \|\bar{x}^i_{2:n_i}\| \geq \frac{\sigma_1}{2k} > 0$ and $\bar{s}^i = 0$; for all $i \in \mathcal{N}$ we have $\bar{s}^i_1 - \|\bar{s}^i_{2:n_i}\| \geq \frac{\sigma_1}{2k} > 0$ and $\bar{x}^i = 0$; and for all $i \in \mathcal{R}$ we have $\bar{x}^i_1 + \bar{s}^i_1 - \|\bar{x}^i_{2:n_i} + \bar{s}^i_{2:n_i}\| \geq \frac{\sigma_2}{2k}$. So we get $\bar{x}^i + \bar{s}^i \in \operatorname{int}(\mathcal{K}^i_q)$ for all $i \in \mathcal{B} \cup \mathcal{N} \cup \mathcal{R}$, which maximize the number of strictly complementary blocks. $\square$

According to Theorem 3.4, we can identify the partition of the four sets $\mathcal{B}$, $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{T}$ as $\mu \to 0_+$. By (3.24)–(3.26), for all $i \in \mathcal{B} \cup \mathcal{N} \cup \mathcal{R}$ we have

$$x^i_1(\mu) + s^i_1(\mu) - \|x^i_{2:n_i}(\mu) + s^i_{2:n_i}(\mu)\| > \min\left\{\frac{\sigma_1}{2k}, \frac{\sigma_2}{2k}\right\},$$

and according to formula (3.27), for all $i \in \mathcal{T}$ we have $x^i_1(\mu) + s^i_1(\mu) - \|x^i_{2:n_i}(\mu) + s^i_{2:n_i}(\mu)\| \to 0$ as $\mu \to 0_+$. Therefore, if we choose μ so small that $x^i_1(\mu) + s^i_1(\mu) - \|x^i_{2:n_i}(\mu) + s^i_{2:n_i}(\mu)\| < \min\{\frac{\sigma_1}{2k}, \frac{\sigma_2}{2k}\}$ for all $i \in \mathcal{T}$, we can separate $\mathcal{T}$ from $\mathcal{B} \cup \mathcal{N} \cup \mathcal{R}$. After that, according to the results of Theorem 3.4, we can separate $\mathcal{B}, \mathcal{N}, \mathcal{R}$ when μ is so small that

$$\frac{k\mu}{\sigma_1} < \min\left\{\frac{\sigma_1}{2k},\ \frac{\sigma_2}{4k}\right\}, \quad \text{and} \quad \max\left\{\frac{k\mu}{\sigma_1},\ \frac{2k\mu}{\sigma_2}\right\} < \frac{\sigma_1}{2k},$$

which is equivalent to

$$\mu < \min\left\{\frac{\sigma_1^2}{2k^2}, \frac{\sigma_1\sigma_2}{4k^2}\right\}.$$

In order to derive bounds for the i^{th} block in the central path with $i \in \mathcal{T}$, we need the following result, which is presented as Theorem 2.4 in [11].

THEOREM 3.6. *For $i = 1, \ldots, m$, let $g_i(x) : R^n \to R$ be quadratic functions. Suppose that the set $\mathcal{S} = \{x \in R^n \mid g_1(x) \leq 0, g_2(x) \leq 0, \ldots, g_m(x) \leq 0\}$ is nonempty. Then for every scalar $\rho > 0$, there exist positive scalars τ and γ such that*

$$\operatorname{dist}(x, \mathcal{S}) \leq \tau \|[g(x)]_+\|^\gamma, \quad \forall x \in R^n \text{ satisfying } \|x\| \leq \rho,$$

where $\operatorname{dist}(x, \mathcal{S})$ is the Euclidean distance from the vector x to the set $\mathcal{S}$, and $[g(x)]_+ = (\max\{g_1(x), 0\}, \max\{g_2(x), 0\}, \ldots, \max\{g_m(x), 0\})$.

Denote the central path as $z(\mu) = (x(\mu), y(\mu), s(\mu))$. By Theorem 3.6, we can get the following estimation for the central path.

THEOREM 3.7. *Suppose $0 < \mu < M$, where M is any positive constant. Then there exist two constants $\tau > 0$ and $\gamma > 0$ such that*

$$\operatorname{dist}(z(\mu), \mathcal{F}^*) \leq \tau \mu^\gamma,$$

where $z(\mu) = (x(\mu), y(\mu), s(\mu))$ is a point on the central path satisfying system (2.5), and $\mathcal{F}^*$ is the set of primal-dual optimal solutions.

Proof. Since the second order cone constraint $x_1 \geq \|x_{2:n}\|$ is equivalent to the following quadratic constraints

$$x_1^2 - \sum_{i=2}^{n} x_i^2 \geq 0 \quad \text{and} \quad x_1 \geq 0.$$

We know that every functions $g_i(z)$, where $z = (x, y, s) \in R^n \times R^m \times R^n$, in systems (2.4) and (2.5) are quadratic, and the solution set of system (2.4) is $\mathcal{F}^*$. By Theorem 2.1 the set $\mathcal{F}^*$ is nonempty. By the convergence and the analyticity properties of the central path $z(\mu)$ in Theorem 2.2, we know that the set $\{z(\mu) \mid 0 < \mu < M\}$ is bounded, i.e., for $0 < \mu < M$ there exists a constant $\rho_M > 0$ such that $\|z(\mu)\| \leq \rho_M$. By system (2.5), where every equality is counted as two inequalities, we have

$$\|[g(x)]_+\| = \sqrt{\sum_{i=1}^{2k} \mu^2} = \sqrt{2k}\mu.$$

Then by Theorem 3.6 we get the desired result. ∎

Using Theorem 3.7, for $i \in \mathcal{T}$ we derive the following estimates for the i^{th} block of variables on the central path.

THEOREM 3.8. *Suppose $0 < \mu < M$, $i \in \mathcal{T}$ and $(x^i(\mu), y^i(\mu), s^i(\mu))$ is the i^{th} block of variables on the central path $(x(\mu), y(\mu), s(\mu))$. Define*

$$\tau_x^i = \max_{(x,y,s)\in\mathcal{F}^*} x_1^i, \quad \tau_s^i = \max_{(x,y,s)\in\mathcal{F}^*} s_1^i.$$

Then there exist constants $\tau_1 > 0$, $\tau_2 > 0$, and $\gamma > 0$ such that

1. *If $\tau_x^i = \tau_s^i = 0$, we have $x^i = s^i = 0$ for $\forall\, (x, y, s) \in \mathcal{F}^*$, and*

$$\tau_2\mu^{1-\gamma} \leq x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| \leq x_1^i(\mu) \leq \tau_1\mu^\gamma,$$

$$\tau_2\mu^{1-\gamma} \leq s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| \leq s_1^i(\mu) \leq \tau_1\mu^\gamma.$$

2. *If $\tau_x^i > 0$, we have $s^i = 0$ for $\forall\, (x, y, s) \in \mathcal{F}^*$, i.e., we have $\tau_s^i = 0$ and*

$$\tau_2\mu^{1-\gamma} \leq x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| \leq \tau_1\mu^\gamma,$$

$$\tau_2\mu \leq s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| \leq \tau_1\mu^\gamma, \quad \tau_2\mu^{1-\gamma} \leq s_1^i(\mu) \leq \tau_1\mu^\gamma.$$

3. *If $\tau_s^i > 0$, we have $x^i = 0$ for $\forall\, (x, y, s) \in \mathcal{F}^*$, i.e., we have $\tau_x^i = 0$ and*

$$\tau_2\mu^{1-\gamma} \leq s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| \leq \tau_1\mu^\gamma,$$

$$\tau_2\mu \leq x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| \leq \tau_1\mu^\gamma, \quad \tau_2\mu^{1-\gamma} \leq x_1^i(\mu) \leq \tau_1\mu^\gamma.$$

Moreover, we have $0 < \gamma \leq \frac{1}{2}$, and there exists a constant $\tau_3 > 0$ such that for all $i \in \mathcal{T}$ we have

$$\tau_2\mu^{1-\gamma} \leq x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\| \leq \tau_3\mu^\gamma.$$

Proof. By Theorem 3.7 there exist constants $\tau > 0$ and $\gamma > 0$ such that

$$\mathrm{dist}(z(\mu), \mathcal{F}^*) \le \tau \mu^\gamma.$$

Since $\mathcal{F}^*$ is compact, there exists $(\bar{x}, \bar{y}, \bar{s}) \in \mathcal{F}^*$ such that

$$\mathrm{dist}(z(\mu), \mathcal{F}^*) = \sqrt{\|x(\mu) - \bar{x}\|^2 + \|y(\mu) - \bar{y}\|^2 + \|s(\mu) - \bar{s}\|^2}.$$

By the above two inequalities we get

$$\|x^i(\mu) - \bar{x}^i\| \le \tau\mu^\gamma, \quad \|s^i(\mu) - \bar{s}^i\| \le \tau\mu^\gamma, \quad \forall i = 1, 2, \ldots, k. \tag{3.28}$$

By the proof of Theorem 3.7 we know that there exists a constant $\rho_M > 0$ such that $\|z(\mu)\| \le \rho_M$ for all $0 < \mu < M$. In the following analysis, we assume $i \in \mathcal{T}$ and let

$$\tau_1 = \sqrt{2}\tau > 0, \ \tau_2 = \min\left\{\frac{1}{3\tau}, \frac{1}{2\rho_M}\right\} > 0.$$

1. If $\tau_x^i = \tau_s^i = 0$, then by the definition of τ_x^i and τ_s^i we have $x_1^i = s_1^i = 0$ for all $(x, y, s) \in \mathcal{F}^*$. Since $x_1^i \ge \|x_{2:n_i}^i\|$ and $s_1^i \ge \|s_{2:n_i}^i\|$, we get $x^i = s^i = 0$ for all $(x, y, s) \in \mathcal{F}^*$. Hence in formula (3.28) we have $\bar{x}^i = \bar{s}^i = 0$, and thus

$$\|x^i(\mu)\| \le \tau\mu^\gamma, \quad \|s^i(\mu)\| \le \tau\mu^\gamma. \tag{3.29}$$

   Then by (3.15) and (3.29) we have

$$x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| \ge \frac{1}{2}\,\frac{\mu}{s_1^i(\mu)} \ge \frac{1}{2}\,\frac{\mu}{\|s^i(\mu)\|} \ge \frac{1}{2\tau}\mu^{1-\gamma}.$$

   Since $\tau_1 = \sqrt{2}\tau > \tau, \ \tau_2 \le \frac{1}{3\tau} < \frac{1}{2\tau}$, by the above formula and (3.29) we get

$$\tau_2\mu^{1-\gamma} \le x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| \le x_1^i(\mu) \le \|x^i(\mu)\| \le \tau_1\mu^\gamma.$$

   In the similar way as above we can get

$$\tau_2\mu^{1-\gamma} \le s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| \le s_1^i(\mu) \le \tau_1\mu^\gamma.$$

2. Suppose $\tau_x^i > 0$. Since $\mathcal{F}^*$ is compact, there exists an $(\hat{x}, \hat{y}, \hat{s}) \in \mathcal{F}^*$ such that $\hat{x}_1^i = \tau_x^i > 0$, $\|\hat{x}_{2:n_i}^i\| = \hat{x}_1^i > 0$ and $\hat{s}^i = 0$ by the definition of $\mathcal{T}$. The proof is by contradiction. If $\tau_s^i \ne 0$, then we have $\tau_s^i > 0$. Therefore there also exist an $(\tilde{x}, \tilde{y}, \tilde{s}) \in \mathcal{F}^*$ such that $\tilde{s}_1^i = \tau_s^i > 0$, $\|\tilde{s}_{2:n_i}^i\| = \tilde{s}_1^i > 0$, and $\tilde{x}^i = 0$. Since $\mathcal{F}^*$ is convex, we have

$$(\breve{x}, \breve{y}, \breve{s}) = \frac{1}{2}(\hat{x}, \hat{y}, \hat{s}) + \frac{1}{2}(\tilde{x}, \tilde{y}, \tilde{s}) \in \mathcal{F}^*.$$

   On the other hand, we have

$$\breve{x}_1^i = \frac{\hat{x}_1^i + \tilde{x}_1^i}{2} = \frac{\hat{x}_1^i}{2} > 0, \ \breve{s}_1^i = \frac{\hat{s}_1^i + \tilde{s}_1^i}{2} = \frac{\tilde{s}_1^i}{2} > 0,$$

   which means $i \in \mathcal{R}$, that is in contradiction with $i \in \mathcal{T}$. Therefore we must have $\tau_s^i = 0$, which means $s^i = 0$ for all $(x, y, s) \in \mathcal{F}^*$. So we have $\bar{s}^i = 0$ in (3.28), and we get

$$\|s^i(\mu)\| \le \tau\mu^\gamma. \tag{3.30}$$

Then by (3.15) and (3.30) we obtain

$$x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| \geq \frac{1}{2}\,\frac{\mu}{s_1^i(\mu)} \geq \frac{1}{2}\,\frac{\mu}{\|s^i(\mu)\|} \geq \frac{1}{2c}\mu^{1-\gamma}. \tag{3.31}$$

Since $\bar{x}_1^i = \|\bar{x}_{2:n_i}^i\|$ for $i \in \mathcal{T}$, by (3.28) we get

$$\begin{aligned} x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| &= (x_1^i(\mu) - \bar{x}_1^i) + (\|\bar{x}_{2:n_i}^i\| - \|x_{2:n_i}^i(\mu)\|) \\ &\leq |x_1^i(\mu) - \bar{x}_1^i| + \|\bar{x}_{2:n_i}^i - x_{2:n_i}^i(\mu)\| \\ &\leq \sqrt{2}\|x^i(\mu) - \bar{x}^i\| \leq \sqrt{2}\tau\mu^\gamma. \end{aligned} \tag{3.32}$$

Then by (3.15) and (3.32) we obtain

$$s_1^i(\mu) \geq \frac{\mu}{x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\|}\,\frac{1}{2} \geq \frac{1}{3c}\mu^{1-\gamma}. \tag{3.33}$$

Symmetrically by (3.23) and $x_1^i(\mu) \leq \|x^i(\mu)\| \leq \|z(\mu)\| \leq \rho_M$ we get:

$$s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| \geq \frac{\mu}{x_1^i(\mu)}\,\frac{1}{2} \geq \frac{1}{2\rho_M}\mu. \tag{3.34}$$

Since $\tau_1 = \sqrt{2}\tau,\ \tau_2 \leq \frac{1}{3\tau} < \frac{1}{2\tau},\ \tau_2 \leq \frac{1}{2\rho_M}$ and $s_1^i(\mu) \leq \|s^i(\mu)\|$, by formulae (3.30)–(3.34) we have

$$\tau_2\mu^{1-\gamma} \leq x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| \leq \tau_1\mu^\gamma,$$

$$\tau_2\mu \leq s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| \leq \tau_1\mu^\gamma, \quad \tau_2\mu^{1-\gamma} \leq s_1^i(\mu) \leq \tau_1\mu^\gamma.$$

3. Symmetrically, by substituting $x^i(\mu)$ for $s^i(\mu)$ and $s^i(\mu)$ for $x^i(\mu)$, respectively, we can derive the desired result in the same way as we did in item 2.

By the results as above, we get $\tau_1\mu^\gamma \geq \tau_2\mu^{1-\gamma}$ for $0 < \mu < M$. Let $\mu \to 0_+$, we get $\gamma \leq 1 - \gamma$. Combined with $\gamma > 0$ we obtain $0 < \gamma \leq \frac{1}{2}$.

According to the results of items 1–3, only three cases may appear for $i \in \mathcal{T}$, i.e., either $\tau_x^i = \tau_s^i = 0$, or $\tau_x^i > 0$ and $\tau_s^i = 0$, or $\tau_x^i = 0$ and $\tau_s^i > 0$. By the results of items 1–3, for any one of the three cases, we always have either $\tau_2\mu^{1-\gamma} \leq x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\|$ or $\tau_2\mu^{1-\gamma} \leq s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\|$. Thus, for all $i \in \mathcal{T}$ we have

$$\begin{aligned} \tau_2\mu^{1-\gamma} &\leq \max\{x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\|, s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\|\} \\ &\leq (x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\|) + (s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\|) \\ &\leq x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\| \end{aligned} \tag{3.35}$$

If $\tau_x^i = \tau_s^i = 0$, we have $x_1^i(\mu) \leq \tau_1\mu^\gamma$ and $s_1^i(\mu) \leq \tau_1\mu^\gamma$. So we get

$$x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\| \leq x_1^i(\mu) + s_1^i(\mu) \leq 2\tau_1\mu^\gamma. \tag{3.36}$$

If $\tau_x^i > 0$ and $\tau_s^i = 0$, we have $x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| \leq \tau_1\mu^\gamma$ and $s_1^i(\mu) \leq \tau_1\mu^\gamma$. So we get

$$\begin{aligned} x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\| \leq& x_1^i(\mu) + s_1^i(\mu) - (\|x_{2:n_i}^i(\mu)\| - \|s_{2:n_i}^i(\mu)\|) \\ =& x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| + s_1^i(\mu) + \|s_{2:n_i}^i(\mu)\| \\ \leq& x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| + 2s_1^i(\mu) \leq 3\tau_1\mu^\gamma. \end{aligned} \tag{3.37}$$

If $\tau_s^i > 0$ and $\tau_x^i = 0$, we have $s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| \le \tau_1\mu^\gamma$ and $x_1^i(\mu) \le \tau_1\mu^\gamma$. So we get

$$\begin{aligned} x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\| \le & x_1^i(\mu) + s_1^i(\mu) - (\|s_{2:n_i}^i(\mu)\| - \|x_{2:n_i}^i(\mu)\|) \\ = & s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| + x_1^i(\mu) + \|x_{2:n_i}^i(\mu)\| \\ \le & s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| + 2x_1^i(\mu) \le 3\tau_1\mu^\gamma. \quad (3.38) \end{aligned}$$

Let $\tau_3 = 3\tau_1$, then by (3.36)–(3.38) we have

$$x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\| \le \tau_3\mu^\gamma, \quad \forall i \in \mathcal{T}.$$

Combining this inequality with formula (3.35) we have the desired result. □

Considering the analysis presented in Theorem 3.8, we can see that those blocks yield the most challenge whose indices are in the set $\mathcal{T}$. Three cases may occur for every block $i \in \mathcal{T}$: either $\tau_x^i = \tau_s^i = 0$, or $\tau_x^i > 0$ and $\tau_s^i = 0$, or $\tau_x^i = 0$ and $\tau_s^i > 0$. In each situations, the block $(x^i(\mu), y^i(\mu), s^i(\mu))$ of the central path with $i \in \mathcal{T}$ has its own properties. There are similarities, but notable differences too.

We summarize the results of Theorem 3.4 and Theorem 3.8 in Table 3.2, where $\Delta_x^i(\mu) = x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\|$, $\Delta_s^i(\mu) = s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\|$, $\Delta_{xs}^i(\mu) = x_1^i(\mu) + s_1^i(\mu) - \|x_{2:n_i}^i(\mu) + s_{2:n_i}^i(\mu)\|$, and $\tau_1,\ \tau_2,\ \tau_3, \gamma$ are positive constants with $0 < \gamma \le \frac{1}{2}$. Cases 1–3 correspond to the three cases "$\tau_x^i = \tau_s^i = 0$", "$\tau_x^i > 0,\ \tau_s^i = 0$," and "$\tau_x^i = 0,\ \tau_s^i > 0$", respectively for $i \in \mathcal{T}$. Observe, that only one case is possible for every block i), and "\" indicates that we do not have enough information for that item.

TABLE 3.2
Local bounds for the central path

	$\mathcal{B}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{T}$		
				Case 1	Case 2	Case 3
$x_1^i(\mu)$	$\ge \frac{\sigma_1}{2k}$	$\le \frac{k\mu}{\sigma_1}$	$\ge \frac{\sigma_2}{4k}$	$\ge \tau_2\mu^{1-\gamma}$ $\le \tau_1\mu^\gamma$	\	$\ge \tau_2\mu^{1-\gamma}$ $\le \tau_1\mu^\gamma$
$s_1^i(\mu)$	$\le \frac{k\mu}{\sigma_1}$	$\ge \frac{\sigma_1}{2k}$	$\ge \frac{\sigma_2}{4k}$	$\ge \tau_2\mu^{1-\gamma}$ $\le \tau_1\mu^\gamma$	$\ge \tau_2\mu^{1-\gamma}$ $\le \tau_1\mu^\gamma$	\
$\Delta_x^i(\mu)$	$\ge \frac{\sigma_1}{2k}$	$\le \frac{k\mu}{\sigma_1}$	$\le \frac{2k\mu}{\sigma_2}$	$\ge \tau_2\mu^{1-\gamma}$ $\le \tau_1\mu^\gamma$	$\ge \tau_2\mu^{1-\gamma}$ $\le \tau_1\mu^\gamma$	$\ge \tau_2\mu$ $\le \tau_1\mu^\gamma$
$\Delta_s^i(\mu)$	$\le \frac{k\mu}{\sigma_1}$	$\ge \frac{\sigma_1}{2k}$	$\le \frac{2k\mu}{\sigma_2}$	$\ge \tau_2\mu^{1-\gamma}$ $\le \tau_1\mu^\gamma$	$\ge \tau_2\mu$ $\le \tau_1\mu^\gamma$	$\ge \tau_2\mu^{1-\gamma}$ $\le \tau_1\mu^\gamma$
$\Delta_{xs}^i(\mu)$	$\ge \frac{\sigma_1}{2k}$	$\ge \frac{\sigma_1}{2k}$	$\ge \frac{\sigma_2}{2k}$	$\tau_2\mu^{1-\gamma} \le \Delta_{xs}^i(\mu) \le \tau_3\mu^\gamma$		

We may look at the results listed in Table 3.2 horizontally or vertically. If we look horizontally, we can see that if μ is so small that

$$\frac{k\mu}{\sigma_1} < \min\left\{\frac{\sigma_1}{2k}, \frac{\sigma_2}{4k}\right\}, \quad \max\left\{\frac{k\mu}{\sigma_1}, \frac{2k\mu}{\sigma_2}\right\} < \frac{\sigma_1}{2k}$$

and

$$\tau_3\mu^\gamma < \min\left\{\frac{\sigma_1}{2k}, \frac{\sigma_2}{2k}\right\},$$

then we can have a complete separation of the blocks of variables. By the above inequalities we get

$$\mu < \min\left\{\frac{\sigma_1^2}{2k^2}, \frac{\sigma_1\sigma_2}{4k^2}, \left\{\min\left\{\frac{\sigma_1}{2k\tau_3}, \frac{\sigma_2}{2k\tau_3}\right\}\right\}^{\frac{1}{\gamma}}\right\}. \tag{3.39}$$

Therefore, if we choose a positive μ such that (3.39) holds, then we can determine the optimal partition $(\mathcal{B}, \mathcal{N}, \mathcal{R}, \mathcal{T})$ for SOCO.

We can see that Table 3.2 is somewhat complicated. The complexity is mainly caused by the set $\mathcal{T}$. In fact, if $\mathcal{T} = \emptyset$ and μ is small enough, we can identify the three sets $\mathcal{B}, \mathcal{N}, \mathcal{R}$ by comparing the results listed in Table 3.2, without using the two condition numbers σ_1 and σ_2 explicitly.

On the other hand, by looking at the results of Table 3.2 vertically, if μ is so small that $\frac{k\mu}{\sigma_1} < \frac{\sigma_1}{2k}$ and $\frac{2k\mu}{\sigma_2} < \frac{\sigma_2}{4k}$, i.e., if

$$\mu < \min\left\{\frac{\sigma_1^2}{2k^2}, \frac{\sigma_2^2}{8k^2}\right\}, \tag{3.40}$$

we have

$$\begin{aligned}
x_1^i(\mu) \geq x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| &\geq \frac{\sigma_1}{2k} \\
&> \frac{k\mu}{\sigma_1} \geq s_1^i(\mu) \geq s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\|, && \forall i \in \mathcal{B} \\
x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| \leq x_1^i(\mu) &\leq \frac{k\mu}{\sigma_1} \\
&< \frac{\sigma_1}{2k} \leq s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| \leq s_1^i(\mu), && \forall i \in \mathcal{N} \\
x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| \leq \frac{2k\mu}{\sigma_2} &< \frac{\sigma_2}{4k} \leq s_1^i(\mu), && \forall i \in \mathcal{R} \\
s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| \leq \frac{2k\mu}{\sigma_2} &< \frac{\sigma_2}{4k} \leq x_1^i(\mu), && \forall i \in \mathcal{R}.
\end{aligned}$$

Therefore, when $\mathcal{T} = \emptyset$ and μ is so small that (3.40) holds, we will have $i \in \mathcal{B}$ if and only if $x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| > s_1^i(\mu)$, which implies $s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| < x_1^i(\mu)$), and $i \in \mathcal{N}$ if and only if $s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| > x_1^i(\mu)$ (which implies $x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| < s_1^i(\mu)$), and $i \in \mathcal{R}$ if and only if both $x_1^i(\mu) - \|x_{2:n_i}^i(\mu)\| < s_1^i(\mu)$ and $s_1^i(\mu) - \|s_{2:n_i}^i(\mu)\| < x_1^i(\mu)$.

However, in practice we may not assume that we can calculate points on the central path exactly. Therefore, in the next section we deal with the case when a point $z = (x, y, s)$ is in the vicinity of the central path $z(\mu) = (x(\mu), y(\mu), s(\mu))$. We show that if a point z is in an appropriate neighborhood of the central path $z(\mu)$ and μ is small enough, then we also have a complete separation of blocks of variables into the four sets $\mathcal{B}, \mathcal{N}, \mathcal{R}$ and $\mathcal{T}$, which constitute the optimal partition.

4. Generalizations for approximate centers. In this section we generalize the results of the previous section to the situation, where a point $z = (x, y, s)$ is in a specific neighborhood of the central path $z(\mu)$. Denote

$$\mathcal{F}^0 = \{z = (x, y, s) \mid (x, y, s) \in \mathcal{F}, x \in \operatorname{int}(K),\ s \in \operatorname{int}(K)\}.$$

On the central path $(x^i)^{\mathrm{T}} s^i = \mu > 0$ and $x_1^i s_{2:n_i}^i + s_1^i x_{2:n_i}^i = 0$ for all $i = 1, \ldots, k$. Therefore the following two parameters are introduced to measure the centrality of a point $z = (x, y, s) \in \mathcal{F}^0$:

$$\delta_c(z) = \frac{\max\limits_{i \in J} (x^i)^{\mathrm{T}} s^i}{\min\limits_{i \in J} (x^i)^{\mathrm{T}} s^i}, \quad \eta_c(z) = \max_{i \in J} \frac{\|x_1^i s_{2:n_i}^i + s_1^i x_{2:n_i}^i\|}{(x^i)^{\mathrm{T}} s^i}, \tag{4.1}$$

where $J = \{1, \ldots, k\}$.

Now we can generalize the results of Theorem 3.4 and Theorem 3.8 to points in the vicinity of the central path.

THEOREM 4.1. *Let $z = (x, y, s) \in \mathcal{F}^0$ and denote $\mu = \frac{\sum_{i=1}^{k} (x^i)^{\mathrm{T}} s^i}{k}$. If $\delta_c(z) \le \tau$ for some $\tau > 1$ and $\eta_c(z) \le \theta$ for some $0 < \theta < 1$, then one has*

1. *For all $i \in \mathcal{B}$, we have*

$$x_1^i \ge x_1^i - \|x_{2:n_i}^i\| > \frac{(1-\theta)\sigma_1}{2k\tau}, \quad s_1^i \le \frac{k\mu}{\sigma_1}.$$

2. *For all $i \in \mathcal{N}$, we have*

$$s_1^i \ge s_1^i - \|s_{2:n_i}^i\| > \frac{(1-\theta)\sigma_1}{2k\tau}, \quad x_1^i \le \frac{k\mu}{\sigma_1}.$$

3. *For all $i \in \mathcal{R}$, we have*

$$x_1^i > \frac{(1-\theta)\sigma_2}{4k\tau}, \quad s_1^i > \frac{(1-\theta)\sigma_2}{4k\tau},$$
$$(x_1^i - \|x_{2:n_i}^i\|) + (s_1^i - \|s_{2:n_i}^i\|) \le \frac{2k\mu}{\sigma_2}.$$

In particular, we have

$$x_1^i - \|x_{2:n_i}^i\| < \frac{2k\mu}{\sigma_2} \quad \text{and} \quad s_1^i - \|s_{2:n_i}^i\| < \frac{2k\mu}{\sigma_2}.$$

4. *For $i \in \mathcal{T}$, let $C > 0$ and $M > 0$ be two positive constants, and define*

$$\mathcal{F}_{M,C} = \{z = (x, y, s) \in \mathcal{F}^0 \mid \exists\ 0 < \mu \le M \text{ such that } \|z - z(\mu)\| \le C\},$$

where $z(\mu)$ is a point on the central path of (2.3). Suppose $z \in \mathcal{F}_{M,C}$, then there exist constants $\tau_1 > 0$, $\tau_2 > 0$ and $\frac{1}{2} \ge \gamma > 0$ such that:

(a) *In case of $\tau_x^i = \tau_s^i = 0$, we have*

$$\frac{1-\theta}{\tau} \tau_2 \mu^{1-\gamma} \le x_1^i - \|x_{2:n_i}^i\| \le x_1^i \le \tau_1 \mu^{\gamma},$$

$$\frac{1-\theta}{\tau} \tau_2 \mu^{1-\gamma} \le s_1^i - \|s_{2:n_i}^i\| \le s_1^i \le \tau_1 \mu^{\gamma}.$$

(b) *In case of $\tau_x^i > 0$ and $\tau_s^i = 0$, we have*

$$\frac{1-\theta}{\tau} \tau_2 \mu^{1-\gamma} \le x_1^i - \|x_{2:n_i}^i\| \le \tau_1 \mu^{\gamma},$$

$$\frac{1-\theta}{\tau} \tau_2 \mu \le s_1^i - \|s_{2:n_i}^i\| \le \tau_1 \mu^{\gamma}, \quad \frac{1-\theta}{\tau} \tau_2 \mu^{1-\gamma} \le s_1^i \le \tau_1 \mu^{\gamma}.$$

(c) In case of $\tau_s^i > 0$ and $\tau_x^i = 0$, we have

$$\frac{1-\theta}{\tau}\tau_2\mu^{1-\gamma} \le s_1^i - \|s_{2:n_i}^i\| \le \tau_1\mu^{\gamma},$$

$$\frac{1-\theta}{\tau}\tau_2\mu \le x_1^i - \|x_{2:n_i}^i\| \le \tau_1\mu^{\gamma}, \quad \frac{1-\theta}{\tau}\tau_2\mu^{1-\gamma} \le x_1^i \le \tau_1\mu^{\gamma}.$$

5. For all $i \in \mathcal{B} \cup \mathcal{N}$ we have

$$x_1^i + s_1^i - \|x_{2:n_i}^i + s_{2:n_i}^i\| > \frac{(1-\theta)\sigma_1}{2k\tau}.$$

For all $i \in \mathcal{R}$ we have

$$x_1^i + s_1^i - \|x_{2:n_i}^i + s_{2:n_i}^i\| > \frac{(1-\theta)^2\sigma_2}{2k\tau}.$$

Finally, there exists a constant $\tau_3 > 0$ such that for all $i \in \mathcal{T}$,

$$\tau_2\mu^{1-\gamma} \le x_1^i + s_1^i - \|x_{2:n_i}^i + s_{2:n_i}^i\| \le \tau_3\mu^{\gamma}.$$

Proof. Let

$$t_i := (x^i)^{\mathrm{T}} s^i \equiv x_1^i s_1^i + (x_{2:n_i}^i)^{\mathrm{T}} s_{2:n_i}^i, \tag{4.2}$$
$$\varepsilon^i := x_1^i s_{2:n_i}^i + s_1^i x_{2:n_i}^i, \tag{4.3}$$
$$\tau_1 = \min\{t_i \mid i = 1, \ldots, k\}, \quad \tau_2 = \max\{t_i \mid i = 1, \ldots, k\}. \tag{4.4}$$

Then, using these quantities and the definition of μ, $\delta_c(z)$ and $\eta_c(z)$, we have

$$0 < \tau_2 \le \tau\tau_1, \quad \|\varepsilon^i\| \le \theta t_i, \ \forall i = 1, \ldots, k, \tag{4.5}$$
$$0 < \tau_1 \le t_i \le \tau_2, \ \forall i = 1, \ldots, k, \tag{4.6}$$
$$\tau_1 \le \mu \le \tau_2, \tag{4.7}$$

where the last inequality follows from the inequalities $k\tau_1 \le k\mu = \sum_{i=1}^{k} t_i \le k\tau_2$. Then, by equation (4.3) we get

$$s_{2:n_i}^i = -\frac{s_1^i}{x_1^i} x_{2:n_i}^i + \frac{\varepsilon^i}{x_1^i} \quad \text{or} \quad x_{2:n_i}^i = -\frac{x_1^i}{s_1^i} s_{2:n_i}^i + \frac{\varepsilon^i}{s_1^i}. \tag{4.8}$$

By substituting (4.8) into (4.2) we have

$$s_1^i = x_1^i \left(\frac{t_i}{(x_1^i)^2 - \|x_{2:n_i}^i\|^2} - \frac{\left(\frac{x_{2:n_i}^i}{x_1^i}\right)^{\mathrm{T}} \varepsilon^i}{(x_1^i)^2 - \|x_{2:n_i}^i\|^2} \right) \tag{4.9}$$

$$\text{or} \quad x_1^i = s_1^i \left(\frac{t_i}{(s_1^i)^2 - \|s_{2:n_i}^i\|^2} - \frac{\left(\frac{s_{2:n_i}^i}{s_1^i}\right)^{\mathrm{T}} \varepsilon^i}{(s_1^i)^2 - \|s_{2:n_i}^i\|^2} \right). \tag{4.10}$$

1. For all $i \in B$, just as in the proof of Theorem 3.4, formulae (3.11)–(3.14) still hold with $x^i(\mu)$ and $s^i(\mu)$ replaced by x^i and s^i, respectively. Because $(x, y, s) \in \mathcal{F}^0$, by definition we have $\sum_{i=1}^{k} (x^i)^{\mathrm{T}} s^i = k\mu$. By (4.9), (4.5), and $x_1^i > \|x_{2:n_i}^i\|$, formula (3.15) is changed into

$$
\begin{aligned}
s_1^i &= x_1^i \left(\frac{t_i}{(x_1^i)^2 - \|x_{2:n_i}^i\|^2} - \frac{\left(\frac{x_{2:n_i}^i}{x_1^i}\right)^{\mathrm{T}} \varepsilon^i}{(x_1^i)^2 - \|x_{2:n_i}^i\|^2} \right) \\
&\geq x_1^i \left(\frac{t_i}{(x_1^i)^2 - \|x_{2:n_i}^i\|^2} - \frac{\theta t_i}{(x_1^i)^2 - \|x_{2:n_i}^i\|^2} \right) \\
&= \frac{(1-\theta)t_i}{x_1^i - \|x_{2:n_i}^i\|} \frac{x_1^i}{x_1^i + \|x_{2:n_i}^i\|} > \frac{(1-\theta)t_i}{x_1^i - \|x_{2:n_i}^i\|} \frac{1}{2}.
\end{aligned} \tag{4.11}
$$

Then by (3.14), where $s_1^i(\mu)$ is replaced by s_1^i, (4.11), and formulae (4.5)–(4.7) we obtain

$$
x_1^i - \|x_{2:n_i}^i\| > \frac{(1-\theta)t_i}{s_1^i} \frac{1}{2} \geq \frac{(1-\theta)\sigma_1 \tau_1}{2k\mu} \geq \frac{(1-\theta)\sigma_1}{2k\tau}.
$$

2. Symmetrically, by respectively substituting x^i by s^i and s^i by x^i we can get the desired result in the same way as above.
3. For all $i \in \mathcal{R}$, formulae (3.16)–(3.19) in the proof of Theorem 3.4 still hold. Therefore, (3.21) also holds with $x^i(\mu)$ and $s^i(\mu)$ replaced by x^i and s^i, respectively. Thus we have

$$
(s_1^i - \|s_{2:n_i}^i\|) + (x_1^i - \|x_{2:n_i}^i\|) \leq \frac{2k\mu}{\sigma_2}.
$$

Then by $s_1^i - \|s_{2:n_i}^i\| > 0$ and $x_1^i - \|x_{2:n_i}^i\| > 0$ we get

$$
x_1^i - \|x_{2:n_i}^i\| < \frac{2k\mu}{\sigma_2} \quad \text{and} \quad s_1^i - \|s_{2:n_i}^i\| < \frac{2k\mu}{\sigma_2}. \tag{4.12}
$$

On the other hand, by (4.2), (4.5), (4.8), $x_1^i > \|x_{2:n_i}^i\|$ and the Cauchy-Schwarz inequality we get

$$
\begin{aligned}
t_i &= (x^i)^{\mathrm{T}} s^i = x_1^i s_1^i + (x_{2:n_i}^i)^{\mathrm{T}} s_{2:n_i}^i \\
&= x_1^i s_1^i - \frac{s_1^i}{x_1^i} \|x_{2:n_i}^i\|^2 + \frac{(\varepsilon^i)^{\mathrm{T}} x_{2:n_i}^i}{x_1^i} \\
&\leq \frac{s_1^i}{x_1^i} ((x_1^i)^2 - \|x_{2:n_i}^i\|^2) + \theta t_i \\
&= s_1^i \frac{x_1^i + \|x_{2:n_i}^i\|}{x_1^i} (x_1^i - \|x_{2:n_i}^i\|) + \theta t_i \\
&\leq 2 s_1^i (x_1^i - \|x_{2:n_i}^i\|) + \theta t_i.
\end{aligned} \tag{4.13}
$$

Then by (4.12), (4.13) and formulae (4.5)–(4.7) we obtain

$$
s_1^i \geq \frac{(1-\theta)t_i}{2(x_1^i - \|x_{2:n_i}^i\|)} > \frac{(1-\theta)\sigma_2 t_i}{4k\mu} \geq \frac{(1-\theta)\sigma_2 \tau_1}{4k\tau_2} \geq \frac{(1-\theta)\sigma_2}{4k\tau}.
$$

Analogously, we can get

$$x_1^i > \frac{(1-\theta)\sigma_2}{4k\tau}.$$

4. For all $i \in \mathcal{T}$, we first show that Theorem 3.7 still holds for $z = (x, y, s) \in \mathcal{F}_{M,C}$. By the definition of $\mathcal{F}_{M,C}$, there exists a $0 < \mu \le M$ such that $\|z - z(\mu)\| \le C$. Therefore, the set of points in he vicinity of the central path $z = (x, y, s) \in \mathcal{F}_{M,C}$ is also bounded by the boundedness of the central path when $0 < \mu \le M$, i.e., there exists $\rho_M > 0$ such that $\|z\| \le \rho_M$. Then by (4.2) and (4.3) we get (where every equality is counted as two inequalities)

$$\|[g(x)]_+\| = \sqrt{\sum_{i=1}^{k} 2(t_i^2 + \|\varepsilon^i\|^2)} \le \sqrt{2(1+\theta^2)\sum_{i=1}^{k} t_i^2} \le \sqrt{2(1+\theta^2)k\mu}.$$

Therefore, by Theorem 3.6 there exist constants $c > 0$ and $\gamma > 0$ such that

$$\mathrm{dist}(z, \mathcal{F}^*) \le \tau\mu^\gamma.$$

Hence, there exists some $(\bar{x}, \bar{y}, \bar{s}) \in \mathcal{F}^*$ such that

$$\|x^i - \bar{x}^i\| \le \tau\mu^\gamma, \quad \|s^i - \bar{s}^i\| \le \tau\mu^\gamma, \quad \forall i = 1, 2, \ldots, k. \tag{4.14}$$

In the following analysis, constants τ_1 and τ_2 are the same as the ones defined in the proof of Theorem 3.8.

(a) In case of $\tau_x^i = \tau_s^i = 0$, we have $x^i = s^i = 0$ for all $(x, y, s) \in \mathcal{F}^*$ as pointed out in the proof of Theorem 3.8. Therefore, in formula (4.14) we have $\bar{x}^i = \bar{s}^i = 0$, and so we get

$$\|x^i\| \le \tau\mu^\gamma, \quad \|s^i\| \le \tau\mu^\gamma. \tag{4.15}$$

Then, by (4.11), (4.15), and formulae (4.5)–(4.7) we obtain

$$x_1^i - \|x_{2:n_i}^i\| \ge \frac{(1-\theta)t_i}{s_1^i}\frac{1}{2} \ge \frac{(1-\theta)\mu^{1-\gamma}\tau_1}{2c\mu} \ge \frac{(1-\theta)}{2c\tau}\mu^{1-\gamma}. \tag{4.16}$$

By (4.10), $s_1^i > \|s_{2:n_i}^i\|$, and (4.5) we get

$$\begin{aligned} x_1^i &= s_1^i \left(\frac{t_i}{(s_1^i)^2 - \|s_{2:n_i}^i\|^2} - \frac{\left(\frac{s_{2:n_i}^i}{s_1^i}\right)^{\mathrm{T}} \varepsilon^i}{(s_1^i)^2 - \|s_{2:n_i}^i\|^2} \right) \\ &\ge s_1^i \left(\frac{t_i}{(s_1^i)^2 - \|s_{2:n_i}^i\|^2} - \frac{\theta t_i}{(s_1^i)^2 - \|s_{2:n_i}^i\|^2} \right) \\ &= \frac{(1-\theta)t_i}{s_1^i - \|s_{2:n_i}^i\|}\frac{s_1^i}{s_1^i + \|s_{2:n_i}^i\|} \ge \frac{(1-\theta)t_i}{s_1^i - \|s_{2:n_i}^i\|}\frac{1}{2}. \end{aligned} \tag{4.17}$$

In the same way, by (4.17), (4.15), and formulae (4.5)–(4.7) we obtain

$$s_1^i - \|s_{2:n_i}^i\| \ge \frac{(1-\theta)t_i}{x_1^i}\frac{1}{2} \ge \frac{(1-\theta)\mu^{1-\gamma}\tau_1}{2c\mu} \ge \frac{(1-\theta)}{2c\tau}\mu^{1-\gamma}. \tag{4.18}$$

Then, by (4.15), (4.16), (4.18), and the definitions of τ_1 and τ_2 we have

$$\frac{1-\theta}{\tau}\tau_2\mu^{1-\gamma} \le x_1^i - \|x_{2:n_i}^i\| \le x_1^i \le \|x^i\| \le \tau_1\mu^\gamma,$$

$$\frac{1-\theta}{\tau}\tau_2\mu^{1-\gamma} \le s_1^i - \|s_{2:n_i}^i\| \le s_1^i \le \|s^i\| \le \tau_1\mu^\gamma.$$

(b) In case of $\tau_x^i > 0$ and $\tau_s^i = 0$, we have $\bar{s}^i = 0$ in (4.14), and we get

$$\|s^i\| \le \tau\mu^\gamma. \tag{4.19}$$

Then in the same way as above we can see that formula (4.16) still holds, and so does formula (3.32), where $x^i(\mu)$ is replaced by x^i, i.e., we have

$$\frac{(1-\theta)}{2c\tau}\mu^{1-\gamma} \le x_1^i - \|x_{2:n_i}^i\| \le \sqrt{2}\tau\mu^\gamma. \tag{4.20}$$

Then by (3.32), (4.11), and formulae (4.5)–(4.7) we get

$$s_1^i > \frac{(1-\theta)t_i}{x_1^i - \|x_{2:n_i}^i\|}\,\frac{1}{2} \ge \frac{(1-\theta)t_i}{2\sqrt{2}\tau\mu^\gamma} \ge \frac{(1-\theta)}{2\sqrt{2}\tau\tau}\mu^{1-\gamma}. \tag{4.21}$$

By $x_1^i \le \|x^i\| \le \|z\| \le \rho_M$, (4.17) and formulae (4.5)–(4.7) we get

$$s_1^i - \|s_{2:n_i}^i\| > \frac{(1-\theta)t_i}{2x_1^i} \ge \frac{(1-\theta)t_i}{2\rho_M} \ge \frac{(1-\theta)\mu}{2\rho_M\tau} \tag{4.22}$$

Thus, by using (4.19)–(4.22) and the definitions of τ_1 and τ_2 we obtain

$$\begin{aligned}
&\frac{1-\theta}{\tau}\tau_2\mu^{1-\gamma} \le x_1^i - \|x_{2:n_i}^i\| \le \tau_1\mu^\gamma,\\
&\frac{1-\theta}{\tau}\tau_2\mu \le s_1^i - \|s_{2:n_i}^i\| \le \|s^i\| \le \tau_1\mu^\gamma,\\
&\frac{1-\theta}{\tau}\tau_2\mu^{1-\gamma} \le s_1^i \le \|s^i\| \le \tau_1\mu^\gamma.
\end{aligned}$$

(c) Symmetrically by substituting x^i for s^i and s^i for x^i, respectively, we can get the desired result in the same way as we do in last item.

By $\gamma > 0$ and $\tau_1\mu^\gamma \ge \frac{1-\theta}{\tau}\tau_2\mu^{1-\gamma}$ for all $0 < \mu < M$, we get $\frac{1}{2} \ge \gamma > 0$.

5. The same way as we in the proof of Theorem 3.4, for all $i \in \mathcal{B} \cup \mathcal{N}$ we have

$$\begin{aligned}
&x_1^i + s_1^i - \|x_{2:n_i}^i + s_{2:n_i}^i\|\\
&\quad \ge x_1^i + s_1^i - (\|x_{2:n_i}^i\| + \|s_{2:n_i}^i\|)\\
&\quad \ge \max\{x_1^i - \|x_{2:n_i}^i\|, s_1^i - \|s_{2:n_i}^i\|\} \ge \frac{(1-\theta)\sigma_1}{2k\tau}.
\end{aligned} \tag{4.23}$$

By (4.2), (4.5), $x_1^i > \|x_{2:n_i}^i\|$, $s_1^i > \|s_{2:n_i}^i\|$, and the Cauchy-Schwarz inequality, we get

$$t_i \le \theta(x^i)^{\mathrm{T}}s^i \le \theta(x_1^i s_1^i + \|x_{2:n_i}^i\|\,\|s_{2:n_i}^i\|) \le 2x_1^i s_1^i, \quad \forall i = 1, \ldots, k. \tag{4.24}$$

For all $i \in \mathcal{R}$, we have two cases: $x_1^i \geq s_1^i$ and $x_1^i < s_1^i$. In the case when $x_1^i \geq s_1^i$, by (4.5), (4.8), (4.24), and the results in item 3 we have

$$\begin{aligned}
& x_1^i + s_1^i - \|x_{2:n_i}^i + s_{2:n_i}^i\| \\
&\quad = x_1^i + s_1^i - \left\| \left(1 - \frac{s_1^i}{x_1^i}\right) x_{2:n_i}^i + \frac{\varepsilon^i}{x_1^i} \right\| \\
&\quad \geq x_1^i + s_1^i - \left|1 - \frac{s_1^i}{x_1^i}\right| x_1^i - \frac{\theta t_i}{x_1^i} \\
&\quad \geq 2s_1^i - 2\theta s_1^i \geq \frac{(1-\theta)^2 \sigma_2}{2k\tau}. \qquad (4.25)
\end{aligned}$$

In the case when $x_1^i < s_1^i$, in the same way as above we get

$$\begin{aligned}
& x_1^i + s_1^i - \|x_{2:n_i}^i + s_{2:n_i}^i\| \\
&\quad = x_1^i + s_1^i - \left\| \left(1 - \frac{x_1^i}{s_1^i}\right) s_{2:n_i}^i + \frac{\varepsilon^i}{s_1^i} \right\| \\
&\quad \geq x_1^i + s_1^i - \left|1 - \frac{x_1^i}{s_1^i}\right| s_1^i - \frac{\theta t_i}{s_1^i} \\
&\quad \geq 2x_1^i - 2\theta x_1^i \geq \frac{(1-\theta)^2 \sigma_2}{2k\tau}. \qquad (4.26)
\end{aligned}$$

Therefore for all $i \in \mathcal{R}$, by (4.25) and (4.26) we have

$$x_1^i + s_1^i - \|x_{2:n_i}^i + s_{2:n_i}^i\| \geq \frac{(1-\theta)^2 \sigma_2}{2k\tau}.$$

For all $i \in \mathcal{T}$, in the same way as in the derivation of (3.35)–(3.38), where $x^i(\mu)$ and $s^i(\mu)$ are replaced by x^i and s^i respectively, we can get

$$\frac{1-\theta}{\tau} \tau_2 \mu^{1-\gamma} \leq x_1^i + s_1^i - \|x_{2:n_i}^i + s_{2:n_i}^i\| \leq \tau_3 \mu^\gamma, \quad \forall i \in \mathcal{T}.$$

$\square$

Now we summarize the results of Theorem 4.1 in Table 4.1, where $\omega_1 = \frac{1-\theta}{\tau}, \omega_2 = \frac{(1-\theta)^2}{\tau}$, and other symbols' meanings are the same as that in Table 3.2, where $x^i(\mu)$ and $s^i(\mu)$ are replaced by x^i and s^i respectively.

The results listed in Table 4.1 imply that if μ is so small that

$$\frac{k\mu}{\sigma_1} < \omega_1 \min\left\{\frac{\sigma_1}{2k}, \frac{\sigma_2}{4k}\right\}, \quad \max\left\{\frac{k\mu}{\sigma_1}, \frac{2k\mu}{\sigma_2}\right\} < \omega_1 \frac{\sigma_1}{2k}$$

and

$$\tau_3 \mu^\gamma < \min\left\{\omega_1 \frac{\sigma_1}{2k}, \omega_2 \frac{\sigma_2}{2k}\right\},$$

then we can have a complete separation of the blocks of variables. Thus, we have

$$\mu < \min\left\{\frac{\omega_1 \sigma_1^2}{2k^2}, \frac{\omega_1 \sigma_1 \sigma_2}{4k^2}, \left\{\min\{\frac{\omega_1 \sigma_1}{2k\tau_3}, \frac{\omega_2 \sigma_2}{2k\tau_3}\}\right\}^{\frac{1}{\gamma}}\right\}. \qquad (4.27)$$

TABLE 4.1
Local bounds in the vicinity of the central path

	$\mathcal{B}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{T}$ Case 1	$\mathcal{T}$ Case 2	$\mathcal{T}$ Case 3
x_1^i	$\geq \omega_1 \frac{\sigma_1}{2k}$	$\leq \frac{k\mu}{\sigma_1}$	$\geq \omega_1 \frac{\sigma_2}{4k}$	$\geq \omega_1\tau_2\mu^{1-\gamma}$ $\leq \tau_1\mu^{\gamma}$	$\backslash$	$\geq \omega_1\tau_2\mu^{1-\gamma}$ $\leq \tau_1\mu^{\gamma}$
s_1^i	$\leq \frac{k\mu}{\sigma_1}$	$\geq \omega_1 \frac{\sigma_1}{2k}$	$\geq \omega_1 \frac{\sigma_2}{4k}$	$\geq \omega_1\tau_2\mu^{1-\gamma}$ $\leq \tau_1\mu^{\gamma}$	$\geq \omega_1\tau_2\mu^{1-\gamma}$ $\leq \tau_1\mu^{\gamma}$	$\backslash$
Δ_x^i	$\geq \omega_1 \frac{\sigma_1}{2k}$	$\leq \frac{k\mu}{\sigma_1}$	$\leq \frac{2k\mu}{\sigma_2}$	$\geq \omega_1\tau_2\mu^{1-\gamma}$ $\leq \tau_1\mu^{\gamma}$	$\geq \omega_1\tau_2\mu^{1-\gamma}$ $\leq \tau_1\mu^{\gamma}$	$\geq \omega_1\tau_2\mu$ $\leq \tau_1\mu^{\gamma}$
Δ_s^i	$\leq \frac{k\mu}{\sigma_1}$	$\geq \omega_1 \frac{\sigma_1}{2k}$	$\leq \frac{2k\mu}{\sigma_2}$	$\geq \omega_1\tau_2\mu^{1-\gamma}$ $\leq \tau_1\mu^{\gamma}$	$\geq \omega_1\tau_2\mu$ $\leq \tau_1\mu^{\gamma}$	$\geq \omega_1\tau_2\mu^{1-\gamma}$ $\leq \tau_1\mu^{\gamma}$
Δ_{xs}^i	$\geq \omega_1 \frac{\sigma_1}{2k}$	$\geq \omega_1 \frac{\sigma_1}{2k}$	$\geq \omega_2 \frac{\sigma_2}{2k}$	$\omega_1\tau_2\mu^{1-\gamma} \leq \Delta_{xs}^i \leq \tau_3\mu^{\gamma}$		

Therefore if we choose a positive μ such that (4.27) holds, then we can identify the optimal partition $(\mathcal{B}, \mathcal{N}, \mathcal{R}, \mathcal{T})$ in the vicinity of of the central path for SOCO.

When $\mathcal{T} = \emptyset$, in the same way as in the previous section, by utilizing the results listed in Table 4.1 vertically, if μ is so small that $\frac{k\mu}{\sigma_1} < \omega_1 \frac{\sigma_1}{2k}$ and $\frac{2k\mu}{\sigma_2} < \omega_1 \frac{\sigma_2}{4k}$, i.e.,

$$\mu < \omega_1 \min\left\{ \frac{\sigma_1^2}{2k^2}, \frac{\sigma_2^2}{8k^2} \right\}, \tag{4.28}$$

we have $i \in \mathcal{B}$ if and only if $x_1^i - \|x_{2:n_i}^i\| > s_1^i$, which implies $s_1^i - \|s_{2:n_i}^i\| < x_1^i$; we have $i \in \mathcal{N}$ if and only if $s_1^i - \|s_{2:n_i}^i\| > x_1^i$, which implies $x_1^i - \|x_{2:n_i}^i\| < s_1^i$; and we have $i \in \mathcal{R}$ if and only if both $x_1^i - \|x_{2:n_i}^i\| < s_1^i$ and $s_1^i - \|s_{2:n_i}^i\| < x_1^i$.

5. Conclusions. In this paper we discuss the identification of the optimal partition $\mathcal{B}$, $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{T}$ for SOCO. By defining two condition numbers, which are positive constants only depending on the SOCO problem itself, we prove that sufficiently close to optimality, the optimal partition can be identified along the central path. Then we generalize the results to the vicinity of central path, i.e., close to optimality we can separate the blocks of variables according to the optimal partition in a neighborhood of the central path. The results in this paper may facilitate to design more efficient algorithms for SOCO. By the polynomial complexity of path-following interior point algorithms for SOCO, we can see that the complexity for finding the optimal partition $\mathcal{B}$, $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{T}$ for SOCO is also polynomial.

Further, the results presented in this paper indicate that the properties of those blocks of variables whose index is in the set $\mathcal{T}$ are the most complicated. The variable blocks with index in $\mathcal{B}$, $\mathcal{N}$ or $\mathcal{R}$ are simpler and easier to analyze. As indicated in Theorem 3.8, three situations may occur for the blocks with index in $\mathcal{T}$,. So far we were unable to give an exact estimation for the convergence order γ for blocks i with $i \in \mathcal{T}$ in Theorem 4.1. This is a challenging question that deserve further studies.

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